International Journal of Recent Advances in Multidisciplinary Topics
Volume 3, Issue 2, February 2022
https://www.ijramt.com | ISSN (Online): 2582-7839

Solution to Single Variable Quintic Equation

Roshan Kumar Singh*
Delivery Manager for a Technology Company, Pune, India

***Abstract*: There are solutions for single variable linear equations, quadratic equations (polynomial equation of degree 2), cubic equations (polynomial equation of degree 3) and quartic equations (polynomial equation of degree 4). But when it comes to quintic equation (polynomial equation of degree 5) there has not been any solution in terms of finding roots of a quintic equation algebraically. This paper forms the basis of solving single variable quintic equation algebraically.**

***Keywords*: Compatible, Degree 5, Existence of solution, Polynomial, Quintic, Roots of the equation, Single variable.**

1. Introduction

From time immemorial humans had a knack of finding the solutions to the problems we encountered. And the problem itself had been complex enough to not have any precedent but a vague possibility of existence of a solution. With the spirit of solving the impossible and establishing precedent for the upcoming generations we have brick by brick and layer by layer become a progressive kind. Had it not been for the spirit and temptation to question the unknown or the unreasonable or the unaccomplished we as a human kind would have lost ourselves in the dogma of mundane establishment, unreal assumptions and awe of illogical explanation. With this very spirit and utmost respect to the whole generation of mathematicians I hereby submit algebraic solution to single variable quintic equation, which till now remained unsolved. There are works such as Abel-Ruffini theorem [1] that shows that there are quintic equations not solvable in radicals; and also works pertaining to previous mistakes in related works [2] and then proving the same thing. In this paper I have focused on efforts to find a solution rather than to deal with correctness or mistakes of previous works or theorems hinting towards non-availability of solutions to equations.

2. Setting the Context, Background and Approach

Let us go back to the basics for a while to set the context. A polynomial with single variable of degree n is represented as follows:

$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$-------- [1] *

Where,

$a_n \neq 0$; and $a_i(i\ ranging\ from\ 0\ to\ n), x \in$ set of real or complex numbers.

A polynomial equation of degree n with single variable is represented as:

$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$ ------- [2]

Solving the above polynomial equation of degree n would mean that for given values of non-zero $a_i(i\ ranging\ from\ 0\ to\ n)$ we are in a position of find the values of x in terms of an expression consisting of given values of a_i(i ranging from 0 to n). Such values of x that satisfy the equation may be called roots of the equation

We have solutions for single variable linear equations, quadratic equations (polynomial equation of degree 2), cubic equations (polynomial equation of degree 3) and quartic equations (polynomial equation of degree 4). But when it comes to quintic equation (polynomial equation of degree 5) there has not been any solution in terms of finding roots of a quintic equation algebraically.

Historically, even before the advent of modern mathematics such as algebra, complex numbers earlier mathematicians developed novel geometric solutions to solve quadratic and even cubic equations. With the advent of modern mathematics we developed generic solutions to the equations. But the basic approach of applying constraint and logic remained the same. I will similarly use concepts of algebra, logic and application of constraints to arrive at solution for quintic equations.

3. Simplifying the Quintic Equation

A quintic equation may be represented as follows:

$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ ------ [3]

For the purpose of arriving at a solution, we set certain constraints and simplify the above form of the quintic equation based on logic.

Since, $a_5 \neq 0$, the above can be written as

$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ ------- [4]

If $e = 0$, then $x = 0$, and the above equation becomes quartic to which solution exists.

Therefore, we put a constraint $e \neq 0$ ------ [5]

Equation [4], can be further simplified by expressing above equation in terms of $x + a/5$. This eliminates need of having degree four (4) term.

$(x + \frac{a}{5} - \frac{a}{5})^5 + a(x + \frac{a}{5} - \frac{a}{5})^4 + b(x + \frac{a}{5} - \frac{a}{5})^3 + c(x + \frac{a}{5} - \frac{a}{5})^2 + d(x + \frac{a}{5} - \frac{a}{5}) + e = 0$ ------- [6

$\left(x + \frac{a}{5}\right)^5 + 10\left(\frac{a}{5}\right)^2\left(x + \frac{a}{5}\right)^3 - 10\left(\frac{a}{5}\right)^3\left(x + \frac{a}{5}\right)^2 +$

*Corresponding author: roshanscorp@gmail.com

$5\left(\frac{a}{5}\right)^4\left(x+\frac{a}{5}\right)+\left(\frac{a}{5}\right)^5-4a\left(\frac{a}{5}\right)\left(x+\frac{a}{5}\right)^3+6a\left(\frac{a}{5}\right)^2\left(x+\frac{a}{5}\right)^2-4a\left(\frac{a}{5}\right)^3\left(x+\frac{a}{5}\right)+a\left(\frac{a}{5}\right)^4+b\left(x+\frac{a}{5}\right)^3-3b\left(\frac{a}{5}\right)\left(x+\frac{a}{5}\right)^2+3b\left(\frac{a}{5}\right)^2\left(x+\frac{a}{5}\right)-b\left(\frac{a}{5}\right)^3+c\left(x+\frac{a}{5}\right)^2-2c\left(\frac{a}{5}\right)\left(x+\frac{a}{5}\right)+c\left(\frac{a}{5}\right)^2+d\left(x+\frac{a}{5}\right)-\frac{da}{5}+e=0$---- [7]

Note the absence of $\left(x+\frac{a}{5}\right)^4$ term in the above equation.

The quintic equation can thus be represented in simplified form as

$x^5+bx^3+cx^2+dx+e=0$ ------- [8]

Retaining the constraint of the last term constant $e\neq 0$

If we solve the generic equation [8] then we can solve any quintic equation.

It is a foregone conclusion that n degree single variable polynomial will have exactly n roots of which none, one or more may be repetitive. Let X be a set of all roots of equation [8] such that each x_i belonging to X, i being natural number ranging from 1 to 5, is a value of x in terms of constants, b, c, d, or e satisfies equation [8].

$x_i \in$ X (set of all roots that satisfies equation [8]), $1 \le i \le 5$, $i \in$ N (set of natural numbers) ------- [9]

4. Solving the Quintic Equation

1) Splitting the quintic equation

Splitting the quintic equation into a cubic and quadratic equation and then trying to arrive at the coefficients by comparison with the quintic equation [8] will help us arrive at a solution.

Consider the following split of the quintic expression into a cubic and a quadratic,

$x^5+bx^3+cx^2+dx+e=(x^3+px^2+qx+r)(x^2+sx+t)$ ------- [10]

$(x^3+px^2+qx+r)(x^2+sx+t)=0$------ [11]

Before we start with the comparison it is important to note that just as x has range of values that satisfies the quintic equation [8]

p,q,r,s and t also has set of values repetitive or unique that would qualify equation [11]. Each of p,q,r,s and t will have not exceeding ten (10) such values. Why ten (10)? That's because we can select three (3) or two (2) roots from set of five (5) in 10 different ways (${}^5C_3$, *or* ${}^5C_2$).

Comparing coefficients of x^4, x^3, x^2, x *and constant* (x^0) we arrive at the following constraints for p,q,r,s and t

Comparing coefficients of x^4,

$p+s=0$ ------- [12]

Comparing coefficients of x^3,

$q+ps+t=b$ ------- [13]

Comparing coefficients of x^2,

$r+sq+pt=c$ ------- [14]

Comparing coefficients of x,

$rs+qt=d$ ------- [15]

Comparing constants,

$rt=e$ ------- [16]

Substituting values of p in terms of s, from equation [12]; of t in terms of r, from equation [16] in equation [13]. Intention is to finally have two (2) variables and two (2) equations. Beginning with substitutions that are easy to identify:

$q-s^2+\frac{e}{r}=b$ ------- [17]

Substituting values of p in terms of s, from equation [12]; of t in terms of r, from equation [16]; of q in terms of s, r, from equation [17] in equations [14] and [15].

Substituting in equation [14],

$r+s\left(b+s^2-\frac{e}{r}\right)-\frac{se}{r}=c$ ------- [18]

Equation [18] becomes,

$s^3+\left(b-\frac{2e}{r}\right)s+r-c=0$------- [19]

Substituting in equation [15],

$rs+\frac{\left(b+s^2-\frac{e}{r}\right)e}{r}=d$ ------- [20]

Equation [20] becomes,

$s^2+\left(\frac{r^2}{e}\right)s+b-\frac{e}{r}-\frac{dr}{e}=0$ ------- [21]

Let's observe equations [19] and [21], they are cubic and quadratic equations of s respectively, for any given specific value of r

For a given value of r there exists at least one s_1 such that it is common root of s for equations [19] and [21]. Therefore, for the given value of r let the roots of s for equation [19] be s_1,s_2,s_3; the roots of s for equation [21] will thus be s_1,s_4.

Restating equations [19] and [21] for quick reference,

$s^3+\left(b-\frac{2e}{r}\right)s+r-c=0$ ------- *[19]*

$s^2+\left(\frac{r^2}{e}\right)s+b-\frac{e}{r}-\frac{dr}{e}=0$ ------- *[21]*

The following equations would result as sum of roots, product of roots of s in equations [19] and [21].

$s_1+s_4=-\frac{r^2}{e}$ ----------- [22]

$s_1+s_2+s_3=0$ ----------- [23]

$s_1s_2=b-\frac{e}{r}-\frac{dr}{e}$ ----------- [24]

$s_1s_2s_3=-(r-c)$ ----------- [25]

2) Determining at least one root of the quintic equation

Assumption 1: There exists 2 (two) common roots of s for a given value of r in equations [19] and [21]

Let s_2 be the other common root. Thus, $s_2=s_4$. ------- [26]

From equations [22] , [23] and [26] we can easily conclude that,

$s_3=\frac{r^2}{e}$ ----------- [27]

From the above equations the following can be derived

Substituting values of s_1,s_2,s_3 from equations [24], [27] in equation [25], we get the following equation in r,

$\frac{r^2}{e}(b-\frac{e}{r}-\frac{dr}{e})=-(r-c)$ ----------- [29]

The above equation becomes,

$dr^3-ber^2+ce^2=0$ ----------- [30]

Note equation [30] is a cubic equation.

Out of curiosity, finding at least one root of the quintic solution

Replacing r with $\frac{e}{t}$ in equation [30] from equation [16], we get cubic equation of t as follows,

$ct^3-bet+de=0$ ----------- [31]

Solving above equation [31] for one of the values of t we get,

$t = y + \frac{be}{3cy}$ where $y = (\frac{-de+\sqrt{d^2e^2-\frac{4b^3e^3}{27c}}}{2c})^{1/3}$ -------- [32]

One of the values of r from equations [16] and [32] will be

$r = \frac{3cey}{3cy^2+be}$ where $y = (\frac{-de+\sqrt{d^2e^2-\frac{4b^3e^3}{27c}}}{2c})^{1/3}$ -------- [33]

One of the values of s from equation [21] and [33]

$s = -\frac{r^2}{2e}(1+\sqrt{1-\frac{4(ber-e^2-dr^2)e}{r^5}})$ where $r = \frac{3cey}{3cy^2+be}$ and $y = (\frac{-de+\sqrt{d^2e^2-\frac{4b^3e^3}{27c}}}{2c})^{1/3}$ --------- [34]

One of the roots of the quintic equation by solving the quadratic equation, $x^2 + sx + t = 0$, will be

$x_1 = \frac{-s+\sqrt{s^2-4t}}{2}$ where, $s = -\frac{r^2}{2e}(1+\sqrt{1-\frac{4(ber-e^2-dr^2)e}{r^5}})$; $r = \frac{3cey}{3cy^2+be}$; $t = y + \frac{be}{3cy}$; $y = (\frac{-de+\sqrt{d^2e^2-\frac{4b^3e^3}{27c}}}{2c})^{1/3}$ ----------- [35

Thus for Assumption 1, there is a solution for the quintic equation.

3) Assumption 2: There exists one and only one common root of s for a given value of r in equations [19] and [21]

If we study the way equations [19] and [21] have been derived we see that equation [19] has been primarily derived from [12], [13], [14] and [16]; while equation [21] has been derived from [12], [13], [15] and [16]. Listed below are the same equations once again for quick reference

$p + s = 0$ ------- [12]

$q + ps + t = b$ ------- [13]

$r + sq + pt = c$ ------- [14]

$rs + qt = d$ ------- [15]

$rt = e$ ------- [16]

$s^3 + \left(b - \frac{2e}{r}\right)s + r - c = 0$ ------- [19]

$s^2 + \left(\frac{r^2}{e}\right)s + b - \frac{e}{r} - \frac{dr}{e} = 0$ ------- [21]

This also means, those roots of s in the above equations for a given value of r that are not common to both the equations will satisfy one of the equations and a parallel equation with a different value of coefficient of x (in short different 'd'', say d_2, d_3 for s_2, s_3) and a different value of coefficient of x^2 (in short different 'c'', say c_4 for s_4). Thus, arriving at following conclusions for non-common roots s_2, s_3, s_4.

There exists d_2 such that s_2 is a common root of s for the given value of r for the following equations:

$s^3 + \left(b - \frac{2e}{r}\right)s + r - c = 0$ ------- [19]

$s^2 + \left(\frac{r^2}{e}\right)s + b - \frac{e}{r} - \frac{d_2r}{e} = 0$ ------- [36]

That ultimately satisfies the equation,

$x^5 + bx^3 + cx^2 + d_2x + e = 0$ ------- [37]

There exists d_3 such that s_3 is a common root of s for the given value of r for the following equations:

$s^3 + \left(b - \frac{2e}{r}\right)s + r - c = 0$ ------- [19]

$s^2 + \left(\frac{r^2}{e}\right)s + b - \frac{e}{r} - \frac{d_3r}{e} = 0$ ------- [38]

That ultimately satisfies the equation,

$x^5 + bx^3 + cx^2 + d_3x + e = 0$ ------- [39]

There exists c_4 such that s_4 is a common root of s for the given value of r for the following equations:

$s^3 + \left(b - \frac{2e}{r}\right)s + r - c_4 = 0$ ------- [40]

$s^2 + \left(\frac{r^2}{e}\right)s + b - \frac{e}{r} - \frac{dr}{e} = 0$ ------- [21]

That ultimately satisfies the equation,

$x^5 + bx^3 + c_2x^2 + dx + e = 0$ ------- [41]

As per the current assumption (Assumption 2), there exists only one common root. Let's refer to equation [30] that is derived under assumption 1($dr^3 - ber^2 + ce^2 = 0$ ----------- [30]).

Under the current assumption (Assumption 2), we can thus safely conclude that $dr^3 - ber^2 + ce^2 \neq 0$. ------- [42]

But we can definitely, say that there exists one and only one unique value ($(ber^2 - ce^2)/r^3$) for a given r that is equal to $d_2 or\ d_3$.

That is,

$d_2 = d_3 = (ber^2 - ce^2)/r^3$ ------- [43]

And, therefore from equations [36], [38] and [43], we can see that s_2, s_3 cater to the same set of equations. Having said that, since s_2, s_3 cater to the same quadratic equation, the sum of which is $(-\frac{r^2}{e})$. Thus,

$s_2 + s_3 = -\frac{r^2}{e}$ ------- [44]

We know from equation [19], sum of roots of the cubic equation [19], $s_1 + s_2 + s_2 = 0$. ------- [45]

Thus from equations [43] and [45] we conclude that

$s_1 = \frac{r^2}{e}$ ------- [46

Substituting the value of $s = \frac{r^2}{e}$ from equation [46] in the quadratic equation [21] we get the following.

$b - \frac{e}{r} - \frac{dr}{e} = 0$ ------- [47]

Equation [48] becomes,

$dr^2 - ber + e^2 = 0$ ------- [48]

Equation [48] is a quadratic equation of r that can be solved to arrive at values of r, t, s, x (one of the roots of the quintic equation) subsequently. Note, that this is valid only for this special scenario of $s = \frac{r^2}{e}$ for a given r in equation [21].

Solving, the above equation we arrive at following value for at least one root of quintic equation, x_1

$r = \frac{be+\sqrt{(be)^2-4de^2}}{2d}$ ------- [49]

$x_1 = \frac{-s+\sqrt{s^2-4t}}{2}$ ------- [50]

Where,

$s = \frac{\left(\frac{be+\sqrt{(be)^2-4de^2}}{2d}\right)^2}{e}$ ------- [51]

$t = \frac{e}{\frac{be+\sqrt{(be)^2-4de^2}}{2d}}$ ------- [52]

Thus we arrive at solution for quintic equation under Assumption 2 as well

Assumption 1 and 2 are complementary to cover all scenarios that may exist and we have solution for both the assumptions, thus solving quintic equation for all scenarios.

5. Conclusion

The above research or mathematical analysis provides solution to quintic equation, going forward there may be other ways as well discovering it through a different approach. While, I would never have succeeded in deriving the solution for quintic solution had there been no prior contribution by revered mathematicians it is important to draw parallels with the approach of earlier mathematicians who arrived at solutions to quadratic, cubic. For quadratic, square was of significance to arrive at a solution. For cubic, it was a unique feature of a cubic that helped arrive at a solution. And for quintic, it is the characteristic that helps it be split into a cubic and quadratic that is of significance while finding a solution. In addition, the approach of transforming the summation of the roots of the original equation to zero is also adopted for arriving at the solution.

References

[1] Abel, N. H. (1824). Mémoire sur les équations algébrique: où on démontre l’impossiblité de la résolution de l’equation générale du cinquième dégré. Librarian, Faculty of Science, University of Oslo.

[2] Pan, Yan & Chen, Yuzhen. (2019). A New Proof of Abel-Ruffini Theorem.