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Solution to Non-Solvable Simple Quintic

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***Abstract*: The paper forms the basis of the solving the famous simple quintic equation, $x^5 - x - 1 = 0$. We will first visit and analyze the assumptions in the paper published [1] 'Solution to Single Variable Quintic Equation' and then pave a way for the solution of the simple quintic equation.**

***Keywords*: Common roots, Equation, First determinant, Galois theory, Quintic, Simple quintic, Single root, Singularity.**

1. Introduction

Solution to Quintic equation is not complete till we have covered all possible scenarios and equations. Mathematicians were enchanted with insolvability of the simple quintic equation, $x^5 - x - 1 = 0$ [2] [3]. After having published [1] solution for quintic equation it will be interesting to see how the concept can be applied to arrive at a solution for the equation $x^5 - x - 1 = 0$.

2. Approach

We will start with the visit to solution for quintic equations [1]. Fix the gaps, refine the solution if required and then apply it to arrive at the solution for simple quintic equation. Proving existence of a solution in this paper would mean coming to a logical conclusion supported by conditions, algebraic equations such that there is a method of arriving at a solution with the help of mechanical calculations there on. The mechanical calculations or the final value itself is not covered in this paper.

Table 1
Solution sets

Cubic roots	p, q, r values	Quadratic roots	s, t values
x_1, x_2, x_3	p_1, q_1, r_1	x_5, x_4	s_1, t_1
x_1, x_2, x_4	p_2, q_2, r_2	x_5, x_3	s_2, t_2
x_1, x_3, x_4	p_3, q_3, r_3	x_5, x_2	s_3, t_3
x_2, x_3, x_4	p_4, q_4, r_4	x_5, x_1	s_4, t_4
x_1, x_2, x_5	p_5, q_5, r_5	x_4, x_3	s_5, t_5
x_1, x_3, x_5	p_6, q_6, r_6	x_4, x_2	s_6, t_6
x_2, x_3, x_5	p_7, q_7, r_7	x_4, x_1	s_7, t_7
x_1, x_4, x_5	p_8, q_8, r_8	x_3, x_2	s_8, t_8
x_2, x_4, x_5	p_9, q_9, r_9	x_3, x_1	s_9, t_9
x_3, x_4, x_5	p_{10}, q_{10}, r_{10}	x_2, x_1	s_{10}, t_{10}

3. Revisit the Solution for Quintic Equation

It is shown in the published paper [1] that a solution to generic equation of the form, $x^5 + bx^3 + cx^2 + dx + e = 0$ can encompass solution to all generic forms of quintic equations.

So revisiting, $x^5 + bx^3 + cx^2 + dx + e = 0$ --------(1)

The above is then expressed in the following form to arrive at the possible solution,

$(x^3 + px^2 + qx + r)(x^2 + sx + t) = 0$ --------(2)

There are 10 possible values of each of p, q, r, s *and* t. This follows from the fact that quintic equation will have 5 roots and there are 10 possible ways to select 3 or 2 roots from it to form a given set of p, q, r, s *and* t. Table 1 illustrates it, x_1, x_2, x_3, x_4, x_5 being the roots of the quintic equation.

p, q, r, s, t may denote any one of the 10 possible sets i.e., p_1, q_1, r_1, s_1, t_1 or p_2, q_2, r_2, s_2, t_2 or p_3, q_3, r_3, s_3, t_3 or…or $p_{10}, q_{10}, r_{10}, s_{10}, t_{10}$.

Let's see some interesting derivations for sets of p, q, r, s, t individually.

$\sum_{i=1}^{10} s_i = -4(x_1 + x_2 + x_3 + x_4 + x_5) = 0$ --------(3)

as we know that sum or roots of (1) is 0 (zero).

$\sum_{i=1}^{10} p_i = -6(x_1 + x_2 + x_3 + x_4 + x_5) = 0$ --------(4

$\sum_{i=1}^{10} q_i = 3\left(\sum_{\substack{i=1,j=1 \\ i\neq j}}^{5} x_i x_j\right) = 3b$ --------(5

$\sum_{i=1}^{10} r_i = -\left(\sum_{\substack{i=1,j=1,k=1 \\ i\neq j\neq k}}^{5} x_i x_j x_k\right) = -(-c) = c$ --------(6

$\sum_{i=1}^{10} t_i = \sum_{\substack{i=1,j=1 \\ i\neq j}}^{5} x_i x_j = b$ --------(7)

Comparing coefficients $x^4, x^3, x^2, x,$ *and* x^0 as we had seen in [1] we had derived the following equations (8) to (14).

Comparing coefficients of x^4,

$p + s = 0$ ------- (8)

Comparing coefficients of x^3,

$q + ps + t = b$ ------- (9)

Comparing coefficients of x^2,

$r + sq + pt = c$ ------- (10)

Comparing coefficients of x,

$rs + qt = d$ ------- (11)

Comparing constants,

$rt = e$ ------- (12)

$s^3 + \left(b - \frac{2e}{r}\right)s + r - c = 0$ ------- (13)

$s^2 + \left(\frac{r^2}{e}\right)s + b - \frac{e}{r} - \frac{dr}{e} = 0$ ------- (14)

1) Assumption 1: 2 common roots

Assumption 1 in [1] can be stated as 'existence of 2 (two)

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common roots of s in equations (13), (14) for any one of the given values of r'. ------- (15)

From Assumption 1 we derived the following equation

$dr^3 - ber^2 + ce^2 = 0$ ----------- (16)

We will call the expression $dr^3 - ber^2 + ce^2$ as first determinant as values of r obtained by equating the expression to 0 (zero) determines the solution for the quintic equation. The determining condition is that there exists a non-zero value of r that satisfies equation (16). ----------- (17)

As Assumption 1 pertains to existence of any one of the given values of r that satisfies equation (16) for the quintic solution to exist, a complementary assumption that would exclude all possibilities in Assumption 1 will thus be a condition of 'existence of one and only one common root of s for all possible values of r in equations (13), (14)

Assumption 2 in [1] should thus be rephrased as 'existence of one and only one common root of s in equations (13), (14) for all possible given values of r at a time'. ----------- (18)

2) Assumption 2: one and only one common root

From Assumption 2 in [1], it was deduced that the common root of s is equal to $\frac{r^2}{e}$.

Assumption 2 is a rarity, because by looking at the common root $s = \frac{r^2}{e}$ and equation (3) we can easily conclude that Assumption 2 is not valid for quintic equations having all real roots. Reason being it is not possible for $\sum \frac{r^2}{e} = 0$ as $r \neq 0$. ----------- (19)

Assumption 2 is thus possible only for quintic equations having at least one complex root.

Following are equations that will lead us to a criteria in terms of coefficients of the quintic equation (1) such that the criteria will be true if Assumption 2 is true for the quintic equation with complex roots.

From equation (3) and common root $s = \frac{r^2}{e}$

$\sum \frac{r^2}{e} = 0$ -------- (20)

From equation (6) we have,

$\sum r = c$

Squaring both sides,

$(\sum r)^2 = c^2$ -------- (21)

Expanding $(\sum r)^2$ in equation (21) we get,

$\sum r^2 + 2(\sum x_i^2 x_j x_k x_l x_m + \sum x_i^2 {x_j}^2 x_l x_m) = c^2$-------- (22)

Note, the second term $\sum x_i^2 x_j x_k x_l x_m = \sum e x_i$ as $x_1 x_2 x_3 x_4 x_5 = e$ thus equation (22) becomes

$\sum r^2 + 2(\sum e x_i + \sum x_i^2 {x_j}^2 x_l x_m) = c^2$-------- (23)

Sum of roots of the quintic equation (1) is 0 (Zero) so $\sum e x_i = 0$; and from equation (20) we know that, $\sum r^2 = 0$

Thus equation (23) becomes,

$\sum x_i^2 {x_j}^2 x_l x_m = \frac{c^2}{2}$ -------- (24

$\sum x_i^2 {x_j}^2 x_l x_m = \sum \frac{x_i^2 {x_j}^2 x_l x_m x_k}{x_k} = \sum \frac{x_i^2 {x_j}^2 x_l x_m x_k}{x_k} = \sum \frac{x_i x_j (x_i x_j x_k x_l x_m)}{x_k} = \sum \frac{x_i x_j (e)}{x_k} = e(\sum \frac{x_i x_j}{x_k})$ -------- (25

From equations (24) and (25) we get,

$\sum \frac{x_i x_j}{x_k} = \frac{c^2}{2e}$ -------- (26)

Note from Table 1, the following holds good,

$\frac{x_4 x_5}{x_1} + \frac{x_4 x_5}{x_2} + \frac{x_4 x_5}{x_3} = \frac{x_4 x_5 (x_1 x_2 + x_1 x_3 + x_2 x_3)}{x_1 x_2 x_3} = -\frac{t_1 q_1}{r_1}$ -------- (27)

From equations (26) and pattern in (27) we conclude that

$\sum \frac{x_i x_j}{x_k} = -\sum (\frac{t_i q_i}{r_i}) = \frac{c^2}{2e}$ -------- (28)

Replacing value of q from equation (11) $q = \frac{d - rs}{t}$, equation (28) becomes,

$-\sum (\frac{t_i (\frac{d - r_i s_i}{t_i})}{r_i}) = \frac{c^2}{2e}$ -------- (29)

$\leftrightarrow \sum \left(s_i - \frac{d}{r_i}\right) = \frac{c^2}{2e}$ -------- (30)

From equation (3), $\sum s_i = 0$, thus equation (30) becomes,

$\sum \left(-\frac{d}{r_i}\right) = \frac{c^2}{2e}$ -------- (31)

From equation (12) and equation (31), the following follows,

$\sum t_i = -\frac{c^2}{2de^2}$ -------- (31)

From equation (7) we know that $\sum t_i = b$, thus we end up with

$b = -\frac{c^2}{2de^2}$ -------- (32)

Thus we conclude that assumption 2 is possible if the quintic equation (1) does not have all real roots and $b = -\frac{c^2}{2de^2}$ -------- (33)

3) Singularities

The above criteria (33) could be one of the criteria for which Assumption 1 may not be true. Let's call such criteria that may lead to Assumption 1 not being true as singularities. A singularity can be defined as any condition, criteria or constraint of existing coefficients (b, c, d, e) in the quintic equation (1) that may lead to first determinant not being equal to zero thus resulting in no solution for a non-zero r.

Analyzing equation (16) involving first determinant we can determine rest of singularities. They are as follows:

- $d = b = 0$ -------- (34)
- $b = c = 0$ -------- (35)
- $d = c = 0$ -------- (36)

(33), (34), (35) and (36) are the only set of singularities for which equation (16) may not work in obtaining a non-zero value for r.

For any other set of conditions, other than the ones mentioned under singularities, and for the given constraint $e \neq 0$ of quintic equation (1) there will exist at least 1 (one) non-zero value of r as a solution to equation (16). This claim is important as it will form the basis of arriving at the solution for the simple quintic equation mentioned below. -------- (37

$x^5 - x - 1 = 0$ -------- (38)

4. Solution to the Simple Quintic Equation

If we examine the simple quintic equation (38) we will immediately recognize that it satisfies one of the singularity conditions (35), i.e. $b = c = 0$.

How do we solve such singularities? One of the novel way of solving a singularity is to transform the quintic equation by known means such that none of the singularity conditions are satisfied.

Let's replace x by y, where $x = \frac{1}{y}$, equation (38) becomes,

$y^5 + y^4 - 1 = 0$ -------- (39)

Now let's transform the equation (39) to get rid of degree 4 term. Let's replace y by z, where $y = z - \frac{1}{5}$, equation (39) becomes,

$(z - \frac{1}{5})^5 + (z - \frac{1}{5})^4 - 1 = 0$ -------- (40)

$\leftrightarrow z^5 + \frac{2}{5}z^3 + \frac{4}{25}z^2 - \frac{3}{125}z - \frac{15621}{15625} = 0$ -------- (41)

Equation (41) is a quintic equation whose sum of roots of z is 0 (zero). And it is also observed that it does not satisfy any of the singularity conditions (33), (34), (35), and (36).

Quintic equation (41) can therefore be solved for at least one of the root of z as illustrated in Assumption 1 in [1] by equating the first determinant to 0 (zero). Thus we can arrive at one of the roots for the simple quintic equation (38) by reverting back value of z by x in the value of the root obtained by replacing z with x from the equation $z = \frac{1}{x} + \frac{1}{5}$. I leave it to enthusiasts to solve the first determinant equation and determine the value of at least one root for the simple quintic equation. I take satisfaction in the fact that I have paved way for finding it out.

We can thus contend by way of example that any quintic equation qualifying for one or more of the singularity conditions can be transformed in such a way so as to make the transformed quintic equation non-singular thus paving the way to solve for at least one of the roots of the original quintic equation.

Needless to say, the above contention also strongly suggests to ignore possibility of solving the Quintic equation on the basis of Assumption 2 except for the check of singularity condition (33) it gives rise to.

In fact the attempt to arrive at a solution under Assumption 2 in [1] except till the point of deduction of the common root of $s = \frac{r^2}{e}$ seems to be incorrect. That is everything till equation ([46]) in [1] is correct. But the equations that follow thereafter from ([47]) onwards are incorrect due to a substitution error.

5. Conclusion

To arrive at definite solution for any quintic equation both the papers, this one and [1] should be looked at in unison. I am not a pundit in Abstract Mathematics but from what I could gather from Abel's proof of insolvability of Quintic equation in terms of radicals [4] [5], this is what I can say – 'Solution for Quintic equation in terms of radicals may not necessarily follow the same pattern as its predecessors the quartic, cubic and quadratic for they are not sufficient enough to prove absence of the contrary'. Any assumption of forms and patterns for radicals to be adopted universally should take into account not just one root as an example but all possible roots in its crudest form with a sound explanation of all possible scenarios. There could be loopholes in the solution that I have presented and I count on the extended community of professionals, mathematicians to review it critically and let me know if it does exist. I will try to address them in my subsequent papers.

References

[1] Roshan Kumar Singh, "Solution to Single Variable Quintic Equation," International Journal of Recent Advances in Multidisciplinary Topics, vol. 3, no. 2, pp. 102-105, 2022.

[2] van der Waerden, Modern Algebra (1949 English edn.), vol. 1, no. 61, pp. 191, 1949.

[3] Lang, Serge Algebraic Number Theory. Graduate Texts in Mathematics. vol. 110, 1994.

[4] Abel, N. H. (1824). Mémoire sur les équations algébrique: où on démontre l'impossiblité de la résolution de l'equation générale du cinquième dégré. Librarian, Faculty of Science, University of Oslo.

[5] Pan, Yan & Chen, Yuzhen. (2019). A New Proof of Abel-Ruffini Theorem.