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Transformation Formalism of Generation of New Exactly Solvable Quantum Potential Systems

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***Abstract*: Extended transformation method is applied to find exactly solvable general quantum mechanical potential system. Exact bound state eigenvalues and Eigen functions of the Schrödinger equation for specific potentials are obtained in any chosen dimensional space, using extended transformation (ET) method which may find applications in different branches of Physics. We have found for multiterm power law potentials, under the framework of ET that a family relationship emerges among the parent and the newly generated exactly solvable potentials (ESPs). The normalizability of bound state solutions of the generated quantum systems is also discussed.**

***Keywords*: Exactly solvable potential, Extended transformation method, Schrödinger equation.**

1. Introduction

Exact analytic solution (EAS) of Schrödinger equation for a given physical quantum system (QS) is desirable as it conveys maximum information of the system. Considerable efforts have been made in recent years towards obtaining exact solution of the Schrödinger equation for potentials of physical interest [1]-[5]. However only a very few potentials governing physical systems yield to analytical solutions. This prompted us to explore and exactly solvable potentials (ESPs) that may exist and which may find applications in different branches of Physics and Chemistry. For this purpose, we have used the extended transformation (ET) method [6]-[9] to generate new ESPs in any desired dimensional space. In this paper we confined ourselves to multiterm potentials only. The extended transformation (ET) includes a coordinate transformation (CT) followed by a functional transformation (FT). A very useful property of the ET method is that the wave functions of the generated quantum systems (QSs) are almost always normalizable provided the behavior of a certain transformation function is smooth.

2. Formalism

The extended transformation method (ET) [6-9] has been applied to generate new exactly solved potentials (ESPs) from an already known ESP.

Let $V_A(r)$ is an exactly solved quantum mechanical central potential in D_A -dimensional space we called as A-QS of the form given by:

$$V_A(r) = \alpha r^a + \beta r^b + \gamma r^c$$

The radial part of the S-wave Schrödinger equation for the potential $V_A(r)$ in D_A -dimensional space $(\hbar = 2m = 1)$:

$$\Psi_A''(r) + \frac{D_A - 1}{r}\Psi_A'(r) + \left[E_n^A - V_A(r) - \frac{l_A(l_A + D_A - 2)}{r^2}\right]\Psi_A(r) = 0$$

The normalized Eigen functions $\Psi_A(r)$ and energy Eigen values E_n^A are already known for the given A-QS for the potential $V_A(r)$.

Under extended transformation (ET), which consist of coordinate transformation $r \to g_B(r)$ and followed by functional transformation:

$$\Psi_B(r) = f^{-1}(r)\Psi_A(g_B(r)) \tag{1}$$

Here the transformation function $g_B(r)$ and the modulated amplitude function $f(r)$ have to be specified within the frame work of ET. As $\Psi_A(r)$ is the Eigen function of exactly solved A-QS potential, hence $\Psi_B(r)$ gets specified exactly, henceforth called B-QS.

The transformed B-QS after implementing ET on A-QS becomes considering $D_A = 3$:

$$\Psi_B''(r) + \left(\frac{d}{dr}\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'}\right)\Psi_B'(r) + \left[\left(\frac{d}{dr}\ln f_B\right)\left(\frac{d}{dr}\ln\frac{f_B' g_B^{D_A-1}}{g_B'}\right) + g_B'^2\left(E_n^A - V_A(g_B) - \frac{l_A(l_A+1)}{g_B^2}\right)\right]\Psi_B(r) = 0$$

Let us choose D_B as the dimension of the B-QS. Then the co-efficient of the term $\Psi_B'(r)$ will take the form:

$$\frac{d}{dr}\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'} = \frac{D_B - 1}{r} = \frac{d}{dr}\ln r^{D_B-1}$$

Integrating,

$$\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'} = \ln r^{D_B-1} - 2\ln N$$

Here $-2\ln N$ is the integration constant.

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This gives:

$$f_B(r) = N g_B^{'\frac{1}{2}} g_B^{-\frac{D_A-1}{2}} r^{\frac{D_B-1}{2}} \tag{2}$$

From equation (1) and (2) we obtain:

$$\Psi_B(r) = g_B^{'-\frac{1}{2}}(r) g_B^{\frac{D_A-1}{2}} r^{-\frac{D_B-1}{r}} \Psi_A(g_B(r)) \tag{3}$$

The corresponding D_B-dimensional expected Schrödinger equation for B-QS is found to be: [6-8]

$$\Psi_B''(r) + \frac{D_B-1}{r}\Psi_B'(r) + \left[E_n^B - V_B(r) - \frac{l_B(l_B+D_B-2)}{r^2}\right]\Psi_B(r) = 0$$

To obtain the above Standard Schrödinger equation form we make the following predefined ansatze [6-9] by taking the first term of A-QS potential as working potential:

$$g_B^{'2}\left(\alpha g_B{}^a(r)\right) = -E_n^B \tag{4}$$

The transformation function is found as:

$$g_B(r) = \left[\frac{\alpha+2}{2}\left(-\frac{E_n^B}{\alpha}\right)^{\frac{1}{2}} r\right]^{\frac{2}{\alpha+2}} \tag{5}$$

And

$$g_B^{'2}(r) E_n^A = -V_B^1 \tag{6}$$

$$-g_B^{'2}(r)\left(V_A(g_B(r)) - \alpha g_B{}^a(r)\right) = -V_B^2 \tag{7}$$

Also

$$\frac{g_B^{'2}\left(l_A+\frac{1}{2}\right)^2}{g_B^2} = \frac{\left(l_B+\frac{D_B}{2}-1\right)^2}{r^2} \tag{8}$$

And

$$\frac{1}{2}\{g_B, r\} = \frac{1}{2}\frac{g_B'''(r)}{g_B'(r)} - \frac{3}{4}\left(\frac{g_B''(r)}{g_B'(r)}\right)^2 = -V_B^3 \tag{9}$$

These ansatz leads the potential of B-QS:

$$V_B(r) = V_B^1(r) + V_B^2(r) + V_B^3(r)$$

Leads to

$$V_B(r) = \alpha_1 r^{a_1} + \beta_1 r^{b_1} + \gamma_1 r^{c_1}$$

With the exponents:

$$a_1 = -\frac{2a}{a+2}, b_1 = \frac{-2a+2c}{a+2}, c_1 = \frac{-2a+2b}{a+2}$$

And parameters:

$$\alpha_1 = C_B^2$$

Where C_B^2 is the Characteristic Constant of B-QS and is:

$$C_B^2 = -E_n^A\left(\frac{a+2}{2}\left(-\frac{E_n^B}{\alpha}\right)^{\frac{1}{2}}\right)^{\frac{4}{a+2}}\left(\frac{2}{a+2}\right)^2$$

And

$$\beta_1 = \left(\frac{a+2}{2}\sigma\right)^{\frac{4+2c}{a+2}}\left(\frac{2}{a+2}\right)^2\gamma$$

$$\gamma_1 = \left(\frac{a+2}{2}\sigma\right)^{\frac{4+2b}{a+2}}\left(\frac{2}{a+2}\right)^2\beta$$

The energy eigenvalue of B-QS is obtained as [6-8]:

$$E_B = -\alpha\left[\frac{C_B^2}{\left(-E_n^A\right)\left(\frac{a+2}{2}\right)^{-\frac{2a}{a+2}}}\right]^{\frac{a+2}{2}}$$

3. Generation of Exactly Solvable quantum Systems

We have considered the following doubly anharmonic potential [1-5]:

$$V_A(r) = \alpha r^6 + \beta r^4 + \gamma r^2$$

With constraint:

$$\frac{\beta^2}{4\alpha} - (2l_A+5)\sqrt{\alpha} = \gamma$$

The normalized Exact energy eigenfunction of the A-QS is:

$$\Psi_A(r) = N_A r^{l_A} \exp\left[-\frac{1}{4}\sqrt{\alpha} r^4 - \frac{1}{4}\frac{\beta}{\sqrt{\alpha}} r^2\right]$$

The energy eigenvalue for the potential system is provided as:

$$E_n^A = \frac{\beta}{\sqrt{\alpha}}\left(l_A + \frac{3}{2}\right).$$

To implement ET, we have selected the working potential term as below:

$$V_A^W(r) = \alpha r^6$$

We obtain the transformation function from equation (6) as:

$g_B(r) = \left[4\left(-\frac{E_n^B}{\alpha}\right)r\right]^{1/4}$ which satisfies the local property $g_B(0) = 0$, which implies that the integration constant is equal to Zero.

Substituting in equation (6), we obtain:

$$V_B{}^1 = C_B^2 r^{-\frac{3}{2}} = \alpha_1 r^{-\frac{3}{2}}$$

Where,

$C_B^2 = \frac{1}{8}\left(-\frac{E_n^B}{\alpha}\right)\left(-E_n^A\right)$ is the Characteristic Constant of B-QS.

From equation (vii), we obtain:

$$V_B{}^2(r) = \beta_1 r^{-1} + \gamma_1 r^{-\frac{1}{2}}$$

Therefore, the potential of B-QS becomes:

$$V_B(r) = \alpha_1 r^{-\frac{3}{2}} + \beta_1 r^{-1} + \gamma_1 r^{-\frac{1}{2}}$$

The constraint equation relating the parameters of the potential and angular momentum quantum number is obtained as:

$$\beta_1 = \frac{4\alpha_1{}^2}{(4l_B + 2D_B - 3)^2} + \frac{\gamma_1}{4\alpha_1}(2l_B + D_B - 1)(4l_B + 2D_B - 3)$$

The energy eigenvalues of B-QS come out as:

$$E_n^B = -\left[\frac{\gamma_1}{4\alpha_1}(4l_B + 2D_B - 3)\right]^2$$

The angular momentum quantum numbers of A-QS and B-QS are related through the relation:

$$8l_B = 2l_A - 4D_B + 9$$

The corresponding normalized energy eigenfunction of B-QS can be read as:

$$\Psi_B(r) = N_B r^{l_B} \exp\left[-\frac{\gamma_1}{4\alpha_1}(4l_B + 2D_B - 3)r + \frac{4\alpha_1}{4l_B + 2D_B - 3} r^{\frac{1}{2}}\right]$$

4. Conclusion

We have presented method of generation of exactly solved quantum system in non-relativistic quantum mechanics using Extended Transformation method in any desired dimensional space. In ET it is possible to generate a number of different exactly solved quantum system by selecting the working potential. We however restrict ourselves by taking only one term working potential. It is also to be noted that the wavefunction of the generated quantum systems are almost always normalizable.

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