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The Mathematics of Tick-Based Timekeeping

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***Abstract*: Timekeeping, via a discrete tick-based system, is a commonly utilized approach for determining the ordered progression of turns, within both the simulation and gaming contexts. Presented in the subject work are the mathematics that underlie such a timekeeping system. This presentation is made within the context of a very popular game, RAID: Shadow Legends. Within this context, the specific instantiations of speed, tick rate, turn meter, speed divisions and critical turn meter value are used to show the transformation of continuous variables into discrete variables with either stepwise continuous values or discrete integer values. When coupled with a simple series of rules that establish the criteria for winning turn meter and which dictate a lack of simultaneous turns, the resultant system is shown to be one in which conflicts at contested ticks are resolved in a pairwise and sequential manner, regardless of the number of combatants involved. Presented in the theory section is the mathematics, based upon modular arithmetic, which is used to determine the relationship between ticks and turns. The solutions are presented in multiple forms, including those based upon using integer sequences and generating functions. Numerical examples are provided to aid in the understanding of the application of the presented theory.**

***Keywords*: game timekeeping, generating functions, integer sequences, modular arithmetic, tick based timing, RAID: Shadow Legends.**

1. Introduction

Axiomatically, it can reasonably be stated that most individuals with the capacity for abstract thinking have a conception of the term time. One may further state that these conceptions, when viewed collectively, do not necessarily tend towards congruence but rather tend towards divergence. This divergence arises, in part, as a function of the contexts within such conceptions are incubated. One may seek to resolve such divergences by turning to a formal evaluative framework, such as the one proposed by Gallie for addressing essentially contested concepts [1], [2]. Alternatively, one may take into consideration lexical definitions of a term under consideration or seek some other approach (singularly or in combination).

In regards to the first approach, the term fails to meet the requirements of the specific framework. Generally, the use of such frameworks far exceeds the necessary considerations of the subject work, except for introductory purposes. In regards to the second approach, one must first note that lexical definitions do not carry the air of authority simply due to their lexical character. The converse of this statement represents an instantiation of the fallacy of (informal) logic of false appeal to (false) authority. With this caveat in place, one finds such definitions as being unappealing secondary to their circular character. As an example, the Oxford English Dictionary [3] defines the term as '*a finite extent or stretch of continued existence, as the interval separating two successive events or actions, or the period during which an action, condition, or state continues*.' Here, the terms finite, continued, interval, successive and period, in the very least, require definitions that are predicated upon the term, time, and thereby render the definition of the term, itself, as circular.

We may readily set aside the Sisyphean task of seeking a ubiquitously applicable definition of the term, time, along with the incumbent vicissitudes of semiotics, by singularly stipulating a definition to the extent that such is needed. For the subject work, we are interested in time as it applies intrinsically to combat in digital games (while not explicitly excluding analog games or simulations) and specifically as a metric for the passage of which leads towards a change in the system manifested by one or more combatants being granted the ability to exercise actions. In such a setting, the interest in time lies within tracking a measure of the same, as a metric, for the purpose of timekeeping.

In a slightly simplified view, timekeeping systems can be classified as either being sufficiently continuous or discrete. The modifier on the first categorical description alludes to the fact that the timekeeping system may actually be discrete but operates in a manner such as to appear to be continuous to a human participant. Such systems may, in turn, be predicated upon 'real time' or an in-game equivalent. For the subject work, such timekeeping systems are not germane. Instead, the context is limited to digital timekeeping. A number of texts on game design provide qualitative information in regards to combat, with general references to timekeeping, but do not provide the details of the mathematics involved with implementing such systems [4]-[14].

For the subject work, the concept of a discrete timekeeping system is done by ensconcing the presentation within the further confines of one specific game. This approach, while requiring some additional introductory information that is game specific, has the potential benefit of crystalizing the presentation within an actual in-use situation. The game chosen for the subject work is RAID: Shadow Legends (RSL), developed by Plarium Global Ltd., a component of the Pixel United subdivision and subsidiary of Aristocrat Leisure Limited. This choice was made due to the following: the entire focus of the game is on combat,

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the combat timekeeping system fits the mechanic of interest, sufficient empirical data exists for parameter estimation, the longevity of the game and the status of the game as the top grossing game in the role-playing game (RPG) category (as per Sensortower.com as of the time of the writing of the subject work).

The game specific background, presented herein, is strictly limited to the points of relevance for the subject work and does not, by any means, serve as a summary of the game, en toto. In brief, the game involves collecting fictional characters (champions) and increasing their efficacy in regards to and by completing game content (all of which either directly involves combat or is directly focused on increasing combat efficacy). The champion archetype is reducible to the following prescribed (immutable to player modification) characteristics: name, faction, rarity, affinity, base combat statistics and skills. The term tribe is sufficiently synonymous with faction. The rarity of a champion fits into one of the categories ranging from common to mythic (a total of six categories). Champion affinity refers to a rock, paper scissors type mechanic coupled with a neutral option. The term combat statistics, while being redundant, refers to eight distinct statistics, of which speed is the most germane. Each champion has a prescribed set of skills, all uniquely named, but all mapping to particular skills from an underlying pool (the skills in the pool can be classified as combat instants, combat buffs or combat debuffs) and most with a requirement of useability once every certain number of turns (i.e. skill cooldown). Any given player can have multiple copies of most champions and any given number of players can have the same champion. While the game has hundreds of such champions, customizability is limited as is any role-playing (the game also lacks any overarching story line).

The game provides two categorical combat scenarios – player verses environment (PVE) and player verses player (PVP). In both cases, a player selects a number of champions, from the pool of champions that they possess, for use in the combat scenario in question. The range for the number of champions that a player may use for any instance of engaging in a combat scenario is dictated by the scenario. The PVE content is prescribed in regards to both the opponents that a player combats, through their champions, and the combat characteristics of those opponents (e.g. a player attempting to defeat the 25th level of the dragon dungeon will always encounter the same computer controlled opponents, with the same statistics and same skills).

The outcome of a combat scenario, for almost all cases, draws from one option of the binary system of winning or losing. In almost all cases, winning is defined by reducing the health of all enemy combatants to zero while having at least one champion with a health amount above zero. The two exceptions to this, both within the scope of PVE content, involve dealing sufficient damage to the computer controlled opponent such as to win the highest level prize from a system that is tiered based upon the damage dealt. The combat system, in all scenarios, is the same. This specifically includes the combat timekeeping system.

Combat timekeeping in the game is based on a tick-based timing system [15-16]. Thusly, the game employs a discrete time, timekeeping, system for combat. If we denote the tick number by the variable name, k, then:

$$k : k \in \mathbb{Z}_0^+ \quad k_{i+1} - k_i = 1 \quad \forall k \tag{1}$$

Stated in the first part of equation (1) is that the ticks are a subset of positive integer values and zero. Stated in the second part of this equation is that the difference between any two successive ticks, for all ticks, is unity (i.e. one). While the discrete tick based timekeeping system is fundamental to combat, the tick count, during any combat scenario, is not visible to the player. The terms ticks and turns are not synonymous. A turn, for a given combatant, is the point in time when they perform an action.

There are two parameters, both visible to any given player, in part, that are salient to determining the ticks at which any given champion is granted a turn. The first is the speed of the champion. This is a continuous variable that can readily be calculated based upon the base speed of the champion, speed granted by equipment, skills, etc. The calculation, itself, is not germane to the subject work. The speed, for any given champion, that is displayed to a player is a rounded integer value of the actual continuous variable (referred to as 'true' speed within the vernacular of the game). The second parameter is the turn meter for each champion. This is displayed, during combat, for each champion, as a bar that visibly fills until a champion is granted a turn, drops to zero when the turn is taken and then starts to refill. This process continues to the completion of the combat scenario. As a variable the turn meter (TM) value is a continuous variable. The displayed value of any given TM has a lower range of zero and an upper range of unity (i.e. 100% normalized to a decimal percentage by dividing by 100%). The actual value of the TM can exceed unity.

There are five general rules that govern the parameters of interest. These are enumerated as follows:

1. Only one combatant is granted a turn at any given time. There are no simultaneous turns.
2. Obtaining a turn at tick k requires winning turn meter at tick k – 1.
3. Turn meter is won by a given combatant when (a) the value of their turn meter is either greater than or equal to unity and (b) the turn meter value for that combatant is greater than the turn meter of all other combatants.
4. For a combatant that wins turn meter at tick k – 1, at tick k, the turn meter for that champion is reduced to zero, that champion takes a turn and then the turn meter for that champion is partially filled.
5. For a combatant that meets the first condition of (3) but not the second, the turn meter for that champion is increased by the applicable amount at tick k – 1, irrespective of whether or not the resultant exceeds unity. The displayed value of the turn meter for that combatant, however, remains at unity.

The rules enumerated above are predicated upon the salient parameters having differing values for all combatants. When the parameters are the same for combatants on opposing teams, the combatant on the team initiating the combat will have preference. When the parameters are the same for combatants on the same team, the order follows the formation order. These are simple deterministic rules and are not considered herein. The tick-based combat timing system, coupled with the enumerated rules, takes the continuous variables of speed and turn meter, and uses them to generate discrete turns. The objective achieved by the subject work is explaining the underlying mathematics of this type of combat system, within the specific instantiation of RSL.

2. Theory

A comprehensive Glossary of Symbols and Terms is provided at the terminus of this work. The first step in the process of developing the underlying theory is relating the continuous variables of speed (s) and turn meter (TM). This is achieved by defining the tick rate (τ), which has empirically [17] been determined to be a linear function of speed. For the ith combatant (the set of all champions of a player within a combat scenario and the opponents), this relationship is defined as (the use of the · symbol denotes standard multiplication rather than a vector dot product):

$$\tau_i = c^{-1} \cdot s_i \tag{2}$$

In equation (2), the parameter c is a constant with a nominal value of the nominal value of 100/0.07. When rounded to six digits of precision, this constant has an approximate value of 1428.57. The apparent range of values for this constant is 1428.56 to 1428.84 [17]. The nominal value is used in calculations in this work. The units of this constant are either (speed · tick) / decimal percentage turn meter or tick / decimal percentage turn meter. The former is predicated on treating the variable, speed, as having units of 'speed' while the latter treats the variable, speed, as being unitless. In either case, the units of tick rate are decimal percentage turn meter per tick. In equation (2), $s_i < c$ for the range of all actual speeds and thusly $\tau_i < 1$ for all actual cases. Finally, since both c and s_i are continuous, τ_i is also continuous.

For any tick rate, we may readily determine the number of ticks that it would take for the corresponding TM to value a minimum value of unity (i.e. one). As an example, if the tick rate was 0.250, it would take four ticks for the corresponding TM to be at unity. More, generally, however, the number of ticks required to meet the first component of the criteria listed in the third rule results in a corresponding TM value that is greater than unity. The number of requisite ticks is clearly a discrete variable (given that it is measured upon discrete ticks) and can be determined by taking the ceiling (i.e. rounding up to the nearest integer value) of the inverse of the tick rate. In the parlance of the game, this is referred to as the speed division (σ). For the ith combatant (the bottom open brackets denote the ceiling operation):

$$\sigma_i = \lceil \tau_i^{-1} \rceil = \lceil c \cdot s_i^{-1} \rceil \tag{3}$$

The mathematical operation shown by equation (3) functions as a method for discretizing speed by taking the integer part of values of a constant divided by different speeds and adding one to the resultant. It should be apparent that the speed division only has positive integer values. Stated succinctly, $\{\sigma_i: \sigma_i \in \mathbb{Z}^+ \ \forall \sigma_i\}$. It should also be apparent that speed values and the corresponding speed divisions are inversely related (i.e. higher speeds map to lower speed divisions and vice versa).

From the last two equations, we may also readily calculate the value of the turn meter that represents the first value that either equals or exceeds unity. We term this the critical turn meter value. For the ith combatant:

$$TM_{ci} = \tau_i \cdot \sigma_i = \tau_i \lceil \tau_i^{-1} \rceil = c^{-1} \cdot s_i \lceil c \cdot s_i^{-1} \rceil \tag{4}$$

Each speed division, except for speed division one, has a specific value of speed that serves as an inclusive lower boundary and a specific speed value that serves as an exclusive upper boundary. For the jth speed division $\{j: j > 1\}$.

$$c \cdot \sigma_j^{-1} \leq s < c\left(\sigma_j - 1\right)^{-1} \tag{5}$$

The exclusive nature of the upper boundary derives from the fact that any given speed only falls into one speed division and that the upper boundary represents the lower boundary for the next speed division. The boundary speed values and boundary critical turn meter values are shown, for speed divisions 2-20, in Table 1 (the lower limits for the critical turn meter are exact while the speeds and upper limits for the critical turn meter are approximate).

Table 1.
Lower (inclusive) and upper (exclusive) speeds and critical turn meter values per speed division for speed divisions 2 – 20

s	s (lower)	s (upper)	TM$_c$ (lower)	TM$_c$ (upper)
2	714.286	1428.571	1	2.000
3	476.190	714.286	1	1.500
4	357.143	476.190	1	1.333
5	285.714	357.143	1	1.250
6	238.095	285.714	1	1.200
7	204.082	238.095	1	1.167
8	178.571	204.082	1	1.143
9	158.730	178.571	1	1.125
10	142.857	158.730	1	1.111
11	129.870	142.857	1	1.100
12	119.048	129.870	1	1.091
13	109.890	119.048	1	1.083
14	102.041	109.890	1	1.077
15	95.238	102.041	1	1.071
16	89.286	95.238	1	1.067
17	84.034	89.286	1	1.063
18	79.365	84.034	1	1.059
19	75.188	79.365	1	1.056
20	71.429	75.188	1	1.053

There are a number of simple observations that one can readily make from Table 1. The first is the range of speeds for any given speed division decreases as the speed division increases. The second is that the lower limit for the critical turn meter, for each speed division, is exactly unity. The third is that for any two differing speed divisions, there exist values of speed for the lower speed division (faster speed) that result in a lower critical turn meter value than the critical turn meter value of a slower combatant (from a higher speed division).

At ticks on which turn meter is contested between the two combatants, the slower combatant would win turn meter.

A. The case of two combatants

The case of two combatants is worthy of consideration, not only in its own right, but also due to the fact that for the case of any N combatants, the determination of combatant preference for ticks on which turns are won is based upon a sequential pairwise comparison. An alphabetic subscript on σ refers to the speed division without consideration of ordering while a numerical subscript refers to the order of the combatant based upon critical turn meter value.

Turn numbers are indicated by the letter n coupled with a double subscript. The first subscript identifies the combatant. The second subscript, when not in parentheses, identifies a specific turn number. Thusly, the notation of n_{ip} means the p^{th} turn of the i^{th} combatant. When the second subscript is within parenthesis, it identifies the turn at which the quantity within the parenthesis is valid. Thusly the notation of $n_{i(a)}$ means the turn at which some quantity, a, occurs or is achieved, for the i^{th} combatant. This subscript system also holds for the case in which the combatant identifier is numerical rather than alphabetic. The domain for all turns is $\{n: n \in \mathbb{Z}_0^+\}$.

All references to the relationship between tick value and turn number are specifically in regards to the tick number for which turn meter is won for the taking the turn. The latter simply occurs at the tick immediately subsequent to the tick on which turn meter was won for taking the turn. The term nominal or the phrase nominal relationship specifically refers to the tick value to turn number relationship, for a combatant, in isolation from all other combatants and prior to any modification. This nominal relationship, for the tick on which the p^{th} turn is won by the i^{th} combatant, is given by the following simple, discrete, linear equation.

$$k = \sigma_i \cdot n_{ip} \tag{6}$$

It is important to note, in equation (6), the independent parameter is the tick number (k) and the dependent parameter is the turn number (n_{ip}). This statement holds due to the fact that the tick number is a common parameter across all combatants. This form can be contrasted against the standard form of y = f(x), where x is the independent parameter, f(x) is a function of the independent parameter and y is the dependent parameter. The subject form was chosen due to the fact that speed divisions appear as integer values rather than as fractional values in the form of $1/\sigma_i$.

Contextually, the speed divisions of interest range from approximately 20, for the slowest speeds, to approximately three, for the maximum theoretical achievable speed. These limits do not impact upon the mathematical development presented in this section but are worthy of note in regards to the context.

1) The case of equal speed divisions

In considering the various pairwise speed division compositions, there is one categorical case in which the actual speeds of the combatants requires direct consideration. That is the case when $\sigma_i = \sigma_j = \sigma$. When both combatants operate within the same speed division, the first contested tick, $k = k_1 = \sigma \cdot n_{i1} = \sigma \cdot n_{i2}$, occurs for the first turn for both combatants. In this case, based upon equation (4), the combatant with the higher speed statistic wins turn meter. Thusly, the nominal relationship of $k = \sigma \cdot n_{1p}\ \forall n_{1p}$ holds for that combatant while the relationship for the opposing combatant changes to $k = \sigma \cdot n_{2p} + 1\ \forall n_{2p}$. There are no subsequent ticks at which turn meter is contested after the first instantiation.

2) Preliminaries for differing speed divisions

The case of equal speed divisions for any pair of combatants is an important case, albeit one that is mathematically trivial. All other cases of interest are predicated upon $\sigma_i \neq \sigma_j$. In this regard, it is useful to consider the relationship between the product of two positive integers, the greatest common divisor (gcd) of the integers and the least common multiple (lcm) of the integers.

$$\sigma_i \cdot \sigma_j = \gcd\left(\sigma_i, \sigma_j\right) \cdot \mathrm{lcm}\left(\sigma_i, \sigma_j\right) \tag{7}$$

The first tick at which turn meter is contested between two combatants, by definition, occurs at the same tick value, for both combatants, albeit at different turn numbers, for each combatant, and with the turn number being a function of the combatant's speed division. If this tick value is denoted as $k = k_1$, it can readily be seen that the following holds.

$$k = k_1 = \mathrm{lcm}\left(\sigma_i, \sigma_j\right) = \frac{\sigma_i \cdot \sigma_j}{\gcd\left(\sigma_i, \sigma_j\right)} \tag{8}$$

The validity of this equation can be seen by the following. When σ_i and σ_j are coprime, $\gcd(\sigma_i, \sigma_j) = 1$, and the first contested tick occurs at a tick number that equals the first multiple of the speed divisions. When σ_i and σ_j are not coprime, they share a common integer divisor, ξ, that is greater than unity. This allows for each speed division to be written as an integer multiple of the common integer divisor. Thusly, $\sigma_i = a_i \cdot \xi$ and $\sigma_j = a_j \cdot \xi$. In the case where there is one common integer divisor that is greater than unity, then by definition, $\xi = \gcd(\sigma_i, \sigma_j)$. If there are more than one common integer divisors that are greater than unity, than ξ can be chosen such that it is the highest valued among all of the common divisors. Substitution into equation (8) leads to the solution of $k = k_1 = \mathrm{lcm}(\sigma_i, \sigma_j) = a_i \cdot a_j \cdot \xi$.

Since the tick number for the first contested tick is the same for both combatants, it is a simple statement that the difference in the value of that tick, as calculated from the speed division and turn number of each combatant, is zero. This can be shown by using equation (6).

$$\sigma_i \cdot n_{i(k_1)} - \sigma_j \cdot n_{j(k_1)} = 0 \tag{9}$$

Equation (9) is a single linear equation with two unknowns, which are the two turn numbers. An equivalent description of this equation is that it is a first order polynomial equation with two unknown integer coefficients. This description fits a standard description of linear Diophantine equation [18]. For such equations, in which the constant term (i.e. the term not multiplied by a polynomial variable) is zero valued, obtaining a solution for the equation can readily be done by simple inspection. Here, the use of equation (8), directly leads to the following solutions.

$$\begin{aligned} n_{i(k_1)} &= \frac{\operatorname{lcm}(\sigma_i, \sigma_j)}{\sigma_i} = \frac{\sigma_j}{\gcd(\sigma_i, \sigma_j)} \\ n_{j(k_1)} &= \frac{\operatorname{lcm}(\sigma_i, \sigma_j)}{\sigma_j} = \frac{\sigma_i}{\gcd(\sigma_i, \sigma_j)} \end{aligned} \tag{10}$$

The adjudication of the conflict at the first contested tick leads to the following results. For the combatant with the higher critical turn meter value, the nominal relationship is retained for all turns (this follows from the rules that were enumerated in the introductory section).

$$k = \sigma_1 \cdot n_{1p} \quad \forall n_{1p} \tag{11}$$

For the combatant with the lower critical turn meter value, the relationship between ticks and turns becomes bifurcated.

$$k = \sigma_2 n_{2q} \quad \left\{ n_{2q} : 1 \le n_{2q} \le n_{2(k_1)} - 1 \right\} \tag{12}$$

$$k = \sigma_2 n_{2q} + 1 \quad \left\{ n_{2q} : n_{2(k_1)} \le n_{2q} \le n_{2(\ )} \right\} \tag{13}$$

In equation (13), the upper limit of the domain has been shown as $n_{2()}$ due to the fact that there may or may not be a mathematically finite upper limit. The solution to this problem is obtained by evaluating the solution at a potential second contested tick. As with the first contested tick, the potential second contested tick would occur when the tick values are the same. Unlike the first contested tick, which was based on the two nominal relationships, the potential second contested tick would involve equations (11) and (13). Letting this potential value be $k = k_2$ it is again noted that the difference in the tick value based upon equations (11) and (13) must be zero. Carrying out this operation followed by algebraic rearrangement leads to the following result.

$$\sigma_1 \cdot n_{1(k_2)} = 1 + \sigma_2 n_{2(k_2)} \tag{14}$$

3) *Speed divisions with common divisors*

The form of equation (14) allows for the introduction of Bezout's identity, which states, in part, that if d is the greatest common divisor of two integers η_1 and η_2 then there exist integers ζ_1 and ζ_2 such that $\eta_1 \cdot \zeta_1 + \eta_2 \cdot \zeta_2 = d = \gcd(\eta_1, \eta_2)$. Here, d, is the minimum integer value and with all other values being integer multiples of d. Comparing this identity to equation (14) shows that there only exists a second contested tick if $\gcd(\sigma_1, \sigma_2) = 1$. This leads to the following two conclusions (both are functionally equivalent). The first is that there only exists contested ticks beyond the first contested tick if σ_1 and σ_2 are coprime. The second conclusion is that if there is a common divisor between σ_1 and σ_2 that is greater than one, then there are no contested ticks beyond the first contested ticks. For the second statement, we can rewrite equation (13) as:

$$k = \sigma_2 n_{2q} + 1 \quad \left\{ \begin{matrix} n_{2q} : n_{2(k_1)} \le n_{2q} \\ \gcd\{\sigma_1, \sigma_2\} > 1 \end{matrix} \right\} \tag{15}$$

4) *Preliminaries for coprime speed divisions*

This leaves one class of binary speed division pairings, that being those that are coprime, for further consideration. For the coprime case, we can readily specify the turns, for each combatant, at which the first conflict occurs.

$$n_{1(k1)} = \sigma_2 \quad n_{2(k1)} = \sigma_1 \tag{16}$$

We proceed by noting that equation (14) can be written as a congruence relationship.

$$\begin{aligned} \sigma_1 \cdot n_{1(k_2)} &= 1 + \sigma_2 n_{2(k_2)} \Leftrightarrow \\ \sigma_1 \cdot n_{1(k_2)} &\equiv 1 \pmod{\sigma_2} \end{aligned} \tag{17}$$

In equation (17), the symbol $\equiv$ denotes congruence and the parentheses mean that mod σ_2 applies to both sides of the equation. There are a number of identities that are useful when it comes to evaluating congruence relationships of this kind. For $\{b_1, b_2, b_3, b_4\} \in \mathbb{Z}$:

$$b_1 + b_3 \equiv b_2 + b_3 \pmod{b_4} \tag{18}$$

$$b_3 \cdot b_1 \equiv b_3 \cdot b_2 \pmod{b_4} \tag{19}$$

$$b_3 \cdot b_1 \equiv b_3 \cdot b_2 \pmod{b_3 \cdot b_4} \tag{20}$$

For $\{b_1, b_2, b_3, b_4, b_5\} \in \mathbb{Z}$ and $\{0 \le b_4 \le b_5\}$:

$$b_1 \cdot b_3 \equiv b_2 \pm b_4 \cdot b_5 \pmod{b_5} \tag{21}$$

For $\{b_1, b_2, b_3, b_4\} \in \mathbb{Z}$, when 'dividing' $b_1 \equiv b_2 \pmod{b_3}$ by b_4, the modulus must be divided by $\gcd(b_3, b_4)$. Finally, we note that for $\{b_1, b_2, b_3\} \in \mathbb{Z}$, $b_1 \cdot b_2 \equiv 1 \pmod{b_3}$ will have positive solutions for the modular multiplicative inverse $b_2^{-1} \pmod{b_3}$ if b_1 and b_3 are relatively prime.

5) *Coprime speed divisions with σ_1 treated as fixed*

For this section we treat σ_1 as fixed in that we are comparing any singular value of σ_1 against multiple values of σ_2. In this regard, we view σ_2 as a first order polynomial (i.e. linear function) of σ_1 where $\{\alpha_{12}, \beta_{12} \in \mathbb{Z}_0^+\}$:

$$\sigma_2 = \alpha_{12}\sigma_1 + \beta_{12} \tag{22}$$

Equation (22) can be substituted into either equation (14) or into the congruence relationship of equation (17).

$$\begin{aligned} \sigma_1 \cdot n_{1(k_2)} &= 1 + (\alpha_{12}\sigma_1 + \beta_{12}) n_{2(k_2)} \Leftrightarrow \\ \sigma_1 \cdot n_{1(k_2)} &\equiv 1 \pmod{\alpha_{12}\sigma_1 + \beta_{12}} \end{aligned} \tag{23}$$

One may readily solve for the turns for each combatant from either equation under (17). However, using equation (23) is more instructive for the work here. Working with the congruence form, we note that we are looking for the minimum valued integer, λ $\{\lambda: \lambda \in \mathbb{Z}^+\}$ such that the multiple of the modulus and λ, followed by the addition of unity, becomes divisible by σ_1.

$$\frac{\lambda\left(\alpha_{12}\sigma_1+\beta_{12}\right)+1}{\sigma_1} \in Z \tag{24}$$

The term in the numerator, $\lambda \cdot \alpha_{12} \cdot \sigma_1$, when divided by σ_1, is clearly integer valued. In order for the entirety of the solution to be an integer, however, the following must also be an integer.

$$\frac{\lambda\beta_{12}+1}{\sigma_1} \in Z \tag{25}$$

The treatment of σ_2 as a linear combination of σ_1 allows for the making of some useful observations when the speed divisions are viewed as simple numbers and when we hold σ_1 fixed. We consider the case where σ_1 is an integer greater than unity and σ_2 is a non-negative integer. Let b and b +1 be two successive integer multiples of σ_1 (i.e. $(\sigma_1 \cdot b)/s_1 = b$ and $(\sigma_1 \cdot (b+1))/\sigma_1 = b+1$). From equation (22), it can be seen that $\beta_{ij} = 0$ when $\sigma_2 = b \cdot \sigma_1$ and when $\sigma_2 = (b + 1) \cdot \sigma_1$. Between these two speed divisions, there are exactly $\sigma_1 - 1$ speed divisions. Furthermore, for these speed divisions, β_{ij} increases by one, starting at one for $\sigma_2 = b \cdot \sigma_1 + 1$, and ending at $\beta_{12} = \sigma_1 - 1$ for $\sigma_2 = (b + 1) \cdot \sigma_1 - 1$. Because b is arbitrary, we can map any {b, b +1} to {0, 1}. This approach works because equation (25) is a function of the remainder β_{12} and not the quotient α_{12}. Furthermore, for any integer λ that satisfies equation (25), the multiple of λ and the quotient will also result in an integer. This observation is useful because (a) it allows for the mapping of problems involving large differences in speed divisions into the simpler problem over the domain, for α_{12}, of {0, 1), (b) it shows that the remainder is limited to values between 1 and $\sigma_1 - 1$, (c) it shows that the determination of λ follows a cyclical pattern in regards to σ_1 and σ_2, and (d) precludes the need for developing separate formulations for $\sigma_1 > \sigma_2$ and $\sigma_1 < \sigma_2$.

When $\beta_{12} = 1$, the solution for λ can trivially be determined to be $\lambda = \sigma_1 - 1$. When $\beta_{12} = \sigma_1 - 1$, the solution for λ can also be determined trivially and is $\lambda = 1$. For the cases where $1 < \beta_{12} < \sigma_1 - 1$, we can rewrite equation (25) by noting that $\beta_{12} = \sigma_1 - (\sigma_1 - \beta_{12})$ and $1 = \sigma_1 - (\sigma_1 - 1)$. Making this substitution followed by algebraic rearrangement leads to the following result.

$$\frac{\lambda\beta_{12}+1}{\sigma_1} = \left(\lambda+1\right) - \frac{\lambda\left(\sigma_1-\beta_{12}\right)+\left(\sigma_1-1\right)}{\sigma_1} \tag{26}$$

If λ is chosen such that the second term on the right of the equality in equation (26) is integer valued, then equation (24) is also integer valued. While this does not provide a singular closed form solution for λ (any integer multiple of λ also produces an integer valued solution), it reduces the problem to one of simple inspection. This holds not only because of the form of the relationship, but also because the solution set for λ, for $1 \leq \beta_{12} \leq \sigma_1 - 1$, maps on a one-to-one basis to integers from the same domain.

It is instructive, at this point, to recall the purpose of finding λ. This parameter was the minimum positive integer that satisfied the requirement of equation (24), which in turn provided the solution for equation (23) which resulted in having the value for the turn, for the combatant with the lower valued critical turn meter, at which the second contested tick occurred. The value of λ is equal to the difference in the number of turns for the combatant with the lower valued critical turn meter for the turns associated with the ticks for the first and second points of contestation.

While this formulation was developed based upon holding σ_1 fixed and writing σ_2 as a linear combination of σ_1 in such a manner as the coefficients of the combination were integers greater than equal to zero, it is apt for any pair $\{\sigma_1, \sigma_2\}$ if α_{12} and β_{12} are defined in a manner consistent with this formulation.

6) Coprime speed divisions with σ_2 treated as fixed

If σ_2 is treated as fixed, an approach that would be apt if one is comparing multiple speed divisions against a known speed division and with the critical turn meter value being greater for all comparisons when paired with the critical turn meter value for the known speed division, it is useful to write the other speed divisions as a linear function of σ_2 where $\{\gamma_{12}, \delta_{12} \in \mathbb{Z}_0^+\}$.

$$\sigma_1 = \gamma_{12}\sigma_2 + \delta_{12} \tag{27}$$

Following the derivation from the previous section, where κ $\{\kappa: \kappa \in \mathbb{Z}^+\}$ is the minimum value for which the following holds:

$$\frac{\kappa\sigma_2+1}{\sigma_1} = \frac{\kappa\sigma_2+1}{\gamma_{12}\sigma_2+\delta_{12}} \in Z \tag{28}$$

Once κ is known, the associated value of λ, as per the previous section, can be solved from dividing the numerator of equation (28) by σ_1. When σ_2 is fixed, there is a pattern present but this pattern, again, is based on λ, which requires an additional calculation, for its determination, than the approach from the previous section. This solution approach is not addressed further given that it can be reduced to the solution approach developed in the previous section.

7) Solution based upon the ceiling function

The previous sections have evaluated the cases of the first two conflicting ticks (i.e. when $k = k_1$ and $k = k_2$). This leads to the obvious questions of whether or not the value of λ informs us as to what happens at a potential third, fourth or any other subsequent point of contestation and how would one account for the tick-turn relationship between these points of contestation. For the first question, we can readily state that the value of λ is applicable for the number of turns between contested ticks, for the combatant with the lower critical turn meter value, for the two combatant case in which the speed divisions are coprime. This statement holds for the following reason: the relationship between ticks and turns for the combatant with the higher critical turn meter value does not change and the resolution at the qth $\{q: q \geq 2, q \in \mathbb{Z}\}$ time at which turns are contested always results in the relationship between ticks and turns for the combatant with the lower valued critical turn meter being incremented by one when compared to the q-1st time at which a contestation arose. One may use this result to write the second equation under equation (13) for each such contestation by having an upper limit as being equal to the lower limit plus λ minus one followed by changing the lower limit to the previous upper limit plus one, changing the upper limit to the new lower limits plus λ minus one and changing the increment to plus two for the next case and so on. Alternatively, one can use a single function that avoids this process. This is done by using the fact that the turns before that at which the first contestation occurs are nominal and by using the ceiling function.

$$k = \sigma_2 n_{2q} + \left\lceil \frac{n_{2q} - \sigma_1 + 1}{\lambda} \right\rceil \quad \{n_{2q} : \sigma_1 \leq n_{2q}\} \tag{29}$$

8) *Solution based upon the mod function*

The ceiling function can be converted to a floor function due to the fact that the numerator and denominator are both positive integers. This conversion relies upon the following identity [19]:

$$\left\lceil \frac{b_6}{b_7} \right\rceil = \left\lfloor \frac{b_6 + b_7 - 1}{b_7} \right\rfloor \quad \{b_6, b_7 \in Z^+\} \tag{30}$$

This conversion avoids having to use a bifurcated form based upon whether or not the operand of the ceiling function is integer valued or not. Employing equation (30) leads to the following:

$$k = \sigma_2 n_{2q} + \left\lfloor \frac{n_{2q} - \sigma_1 + \lambda}{\lambda} \right\rfloor \quad \{n_{2q} : \sigma_1 \leq n_{2q}\} \tag{31}$$

The utility of this conversion from ceiling to floor allows us to employ another identity.

$$\begin{gathered} b_8 \bmod b_9 = b_8 - b_9 \left\lfloor \frac{b_8}{b_9} \right\rfloor \rightarrow \\ \left\lfloor \frac{b_8}{b_9} \right\rfloor = \frac{1}{b_9}\left(b_8 - b_8 \bmod b_9\right) \end{gathered} \quad \{b_8, b_9 \in Z^+\} \tag{32}$$

Here, b_8 mod b_9 denotes the remainder after b_8 is divided by b_9. It does not apply to both sides of the equation and b_8 cannot be factored from it, in the form shown. Applying this equation to the previous equation leads to the following:

$$\begin{aligned} k &= \sigma_2 n_{2q} \\ &+ \frac{1}{\lambda}\begin{pmatrix} n_{2q} - \sigma_1 + \lambda - \\ \left(n_{2q} - \sigma_1 + \lambda\right) \bmod \lambda \end{pmatrix} \end{aligned} \quad \{n_{2q} : \sigma_1 \leq n_{2q}\} \tag{33}$$

Another point, worthy of note, relates to the two special cases of $\beta_{12} = 1$, which leads to the result of $\lambda = \sigma_1 - 1$, and of $\beta_{12} = \sigma_1 - 1$, which leads to the result of $\lambda = 1$. These are special cases because the value of λ is fixed at a known value for both cases. For the second case and substituting $\lambda = 1$ into equation (29) leads to the following result:

$$k = (\sigma_2 + 1) n_{2q} - \sigma_1 + 1 \quad \{n_{2q} : \sigma_1 \leq n_{2q}\} \tag{34}$$

Equation (34) holds for cases when $\sigma_2 + 1$ is equal to an integer multiple of σ_1.

9) *Solution based upon sequence and generating function*

Both equations (29) and (33) are alternative representations of the same result and both require the determination of the parameter λ. Presented in this subsection is another method (that again relies on the determination of λ). The parameter λ, as indicated previously, indicates the number of turns that pass between any two consecutive integer differences, exclusive, when compared to the nominal case, for the situation in which two speed divisions are coprime and the relationship between the ticks on which turns are won and the tick number are modified for the combatant with the lower valued critical turn meter. This parameter is also equal to the number of turns over which a given integer difference holds.

For the combatant with the lower valued critical turn meter, the difference between the modified relationship regarding the tick number on which turn n_{2q} is won when compared to the nominal case is equal to zero for turns one to $n_{2q} = \sigma_1 - 1$. The difference then changes to unity for λ turns, to two for another λ turns and so on. While this analysis can readily start with $n_{2q} = 1$, it is more useful to start the analysis at the first turn at which $\lambda = 1$ and simply substitute at the terminus of the analysis. The difference in turn numbers, for the two cases, thus becomes $\langle 1, \ldots$ repeated $\lambda - 1$ times $\ldots, 2, \ldots$ repeated $\lambda - 1$ times $\ldots, 3, \ldots\rangle$. This sequence can be related to the integer coefficients of an infinite series such as the power series (the ordinary generating function).

$$\begin{aligned} A(x) &= \sum_{r=0}^{\infty} a_r x^r \\ &= a_0 + a_1 x + a_2 x^2 + \cdots \end{aligned} \tag{35}$$

The domain of x in equation (35) is $|x| < 1$. This, however, is immaterial due to the fact that there are no numerical values substituted into x. Instead it is the coefficient values that are of relevance because they are used to encode the values of the difference in the turn numbers. The encoding of unity, for all coefficient values, for example, leads to the following result:

$$\langle 1,1,1,\ldots \rangle \leftrightarrow 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{36}$$

The formula shown in equation (36) can readily be modified in order to generate a different sequence. One manner of modification is by multiplying either equation (35) or the series on the right of the mapping in equation (36) by x. Using the latter, it can readily be seen that the modified generating function is right-shifted due to the fact that all coefficient terms in the original function are now associated with a +1st power of x. Because the original constant term (i.e. term associated with

x^0) is now associated with x^1, the constant term in the modified generating function is zero.

$$\langle 0,1,1,1,\ldots\rangle \leftrightarrow x + x^2 + x^3 + \cdots = \frac{x}{1-x} \tag{37}$$

One may readily right shift the original generating function, through the introduction of any number of leading zeroes, simply by multiplying the original generating function by x raised to a power equal to the number of leading zeroes that one wishes to introduce. For the subject case, the sequence will always be known due to the fact that the calculation of λ is requisite for its use. Let A(x) denote the series consisting of the sequence of positive integers, each repeated λ times, starting with one and subtract from it the series x·A(x), the resultant sequence will be of the form <1, 0 repeated λ – 1 times, 1, 0 repeated λ – 1 times, …>. The generating function for such a sequence is $(1-x^{\lambda})^{-1}$. This results in the following:

$$\begin{aligned} A(x) - x\cdot A(x) &= \frac{1}{1-x^{\lambda}} \rightarrow \\ A(x) &= \frac{1}{(1-x)(1-x^{\lambda})} \end{aligned} \tag{38}$$

Equation (38) can be rewritten by means of a partial fraction expansion, a common technique employed with continuous time integral transforms such as the Laplace transform, or its discrete analog, the Z-transform. The resultant is the sum of ratios of polynomials in x divided by x minus the appropriate root in the expansion. When the form of the partial fraction expansion, on a termwise basis, takes the form shown by equation (39), where $\{\gamma_1, \gamma_2: \gamma_2 > \gamma_1 \geq 0, \gamma_1, \gamma_2 \in \mathbb{N}\}$, then for any $n \geq \gamma_1$, the coefficient of x^n in equation (39) is given by equation (40).

$$\frac{\gamma_3 x^{\gamma_1}}{(1-\gamma_4 x)^{\gamma_2}} \tag{39}$$

$$\frac{\gamma_3 (n-\gamma_1+\gamma_2-1)!\,\gamma_4^{\,n-\gamma_1}}{(n-\gamma_1)!(\gamma_2-1)!} \tag{40}$$

Once carried out, the resultant solution for A(x) is in the form of a recursive relationship. Finally, one may replace the variable n by $n_{2q} - (\sigma_1 - 1)$ to account for the initial zeroes that are generated by the turns prior to the first turn at which turn meter is contested.

B. The case of more than two combatants

The fact that the critical turn meter serves as a driving predicate for the ticks on which turns are won should be readily apparent from the previous section that details the solution for two combatants.This becomes even more apparent when the number of combatants is increased. When the number of combatants is greater than one, one combatant will have precedence at all ticks at which winning turn meter is in conflict with respect to all other combatants. This combatant can be viewed as the combatant that is the anchor for the entire combat scenario. When all combatants have differing speeds, the combatant that serves as the anchor is the one with the highest critical turn meter value. It should be noted that this is not synonymous with the combatant with the highest speed. In the case where a combatant with the highest speed is the anchor, this occurs not because of having the highest speed but rather because of having the greatest value of critical turn meter.

For this section, for $N \geq 3$ combatants, albeit also applicable to $N \geq 2$ combatants, we order the combatants in order of critical turn meter value. Thusly $TM_{c1} > TM_{c2}, \ldots, > TM_{c(N-1)} > TM_{cN}$. This ordering is not predicated upon any particular ordering of the speeds or speed divisions for the combatants. Three conditions ensured that for the case of N = 2 that each numerical, integer, value of k either mapped to no combatant winning turn meter or only one combatant winning turn meter (and in a manner consistent with the rules). These are the nominal relationship for the anchor (turn n_{1j} at $k = \sigma_1 n_{1j}\ \forall n_{1j}$), the nominal relationship for the second combatant prior to the first tick at which turn meter is contested and the appropriate relationship for the second combatant for turns after the turn associated with the fist tick at which turn meter is contested.

When a third combatant is introduced, these three constraint equations take precedence to the nominal relationship between the ticks at which turns would be won and the turn number for the third combatant. It is again important to remember that the tick number, k, is the independent parameter. If σ_3 is equal to either of the two extant speed divisions, or both, the first case described in the previous section, applies. If σ_3 does not match the extant speed divisions then the first constraint encountered will be that which is presented by $\min\{\sigma_1, \sigma_2\}$ and at a value of k being equal to that minimum multiplied by σ_3. The relationship between the tick at which the appropriate turn is won and the turn number for the third combatant is then modified as per the rules noted in the previous section. This modified relationship, rather than the nominal has to then be checked against the constraint imposed by other speed division as well against the form of the constraint associated with $\min\{\sigma_1, \sigma_2\}$. These checks have to be made at each point where the form of the equations change. Thusly, the constraints operate in a sequential manner rather than a simultaneous manner. This renders approaches such as applying the Chinese Remainder Theorem, for simultaneously solving congruence relationships, inappropriate.

As the number of combatants is increased, the corresponding number of extant constraint conditions is increased. The same approach as detailed in the previous paragraph still applies but the number of checks increases with each additional combatant added.

3. Examples

For this section, we consider, first, some numerical examples that exemplify certain aspects that were developed in the previous section. This is followed by two examples, both derived from specific in-game content and both involving PVE content in which winning against the game controlled opponent(s) requires dealing a certain amount of damage such as to cross the threshold for the top tier reward.

A. Examples from theory

For the first example we consider the paired speed divisions $\{\sigma_i, \sigma_j\} = \{2, 4\}$, $\{3, 6\}$, $\{3, 9\}$ and $\{6, 8\}$. For each pair, the combatant operating at σ_i is referred to as the first combatant while the combatant operating at σ_j is referred to as the second combatant.

The results for each pairwise comparison are shown in Table 2. For each case, the tick value at which the first contestation occurs is the least common multiple. For each case, the corresponding turns for each combatant is equal to the tick value divided by the speed division of the opposing partner. For the first case, the modified relationship for the second combatant switches from even to odd. For the third case, the modified relationship for the second combatant switches from odd to even. For these two cases, the relationship for the first combatant remains even and odd respectively. Thusly there are no contested ticks beyond the first. For the second case, when the first combatant wins, the tick turn relationship alternates from odd to even while the relationship for the second combatant switches to odd. For each closest turn pair following the turn pair at the first contested tick, the difference for the second combatant is always +1. For the opposing case (i.e. for the second combatant winning at the first contested tick), the closest turn pair becomes +1 for the first combatant. This same turn differential also occurs for the fourth case. In none of these cases, as expected, is there a contested tick after the first contested tick.

Table 2
Results of the pairwise comparisons between $\{\sigma_1, \sigma_2\}$ showing the greatest common divisor (gcd), least common multiple (lcm), the combatant number with the highest critical turn meter (TM_c), the value of k at which the first turn meter contestation occurs, the associated turn for the second combatant, and the resultant tick-turn modification for the same

{s_i, s_j}	**gcd**	**lcm**	**TM_c**	**k_1**	**n_{2k_1}**	**$n_{2j} \geq n_{2k_1}$**
{2, 4}	2	4	1	4	1	$4n_{2j} + 1$
			2		2	$2n_{2j} + 1$
{3, 6}	3	6	1	6	1	$6n_{2j} + 1$
			2		2	$3n_{2j} + 1$
{3, 9}	3	9	1	9	1	$9n_{2j} + 1$
			2		3	$3n_{2j} + 1$
{6, 8}	2	24	1	24	3	$8n_{2j} + 1$
			2		4	$6n_{2j} + 1$

For the second example, we consider, arbitrarily chosen, a speed division of five and evaluate it as a number when compared to the integers zero through 15 (the contextual applicability of speed divisions starts with two) in regards to the linear relationship given by equation (22) along with the determination of λ. These results are shown in Table 3.

In looking at Table 3, it can readily be seen that the first instantiation of a new integer value for α_{12}, with respect to the numerically increasing value of σ_2, occurs when σ_2 is that integer multiple of σ_1. That value then hold apt for the next four speed divisions (for a total of five speed divisions). At the first instantiation of each integer value, as expected, β_{12} is zero valued. This value increases by one for each speed division, from the instantiating speed division, reaching the expected value of σ_1 -1 = 4 at the speed division immediately prior to the one for each α_{12} first undergoes incrementing by unity. This cyclical pattern is readily apparent.

Table 3
Values for the linear relationship coefficients α_{12} and β_{12} as well as l, where σ_1 = 5 and σ_2 (as a numerical value) as listed

s_2	**a_{12}**	**b_{12}**	**l**
0	0	0	NA
1	0	1	4
2	0	2	2
3	0	3	3
4	0	4	1
5	1	0	NA
6	1	1	4
7	1	2	2
8	1	3	3
9	1	4	1
10	2	0	NA
11	2	1	4
12	2	2	2
13	2	3	3
14	2	4	1
15	3	0	NA

The value of λ at integer multiples of σ_1 is not utilized due to the fact that the theoretical development that used λ was predicated upon coprime speed divisions. The value of λ for each speed division following the one for which σ_2 is an integer multiple of σ_1 has the expected value of $\lambda = \sigma_1 - 1$. The values of σ_2 of $\{1, 6, 11\}$, multiplied by four, followed by the addition of unity results in $\{5, 25, 45\}$. Each value is the first value that is divisible by both $\sigma_1 = 5$ and the corresponding value of σ_2. For the last speed division prior to an increment by unity of α_{12}, the value of λ is the expected value of $\lambda = 1$. The values of σ_2 of $\{4, 9, 14\}$, multiplied by one, followed by the addition of unity results in $\{5, 10, 15\}$. Each value is again the first value that is divisible by both $\sigma_1 = 5$ and the corresponding value of σ_2. The value of $\lambda = 2$ or $\lambda = 3$ was based upon knowing that they were the only two remaining values in the integer set from one to $\sigma_1 - 1$ and by simple inspection. For the values of σ_2 of $\{2, 7, 12\}$, multiplication by $\lambda = 2$, plus unity leads to the result of $\{5, 15, 25\}$. For the values of $\sigma_2 = \{3, 8, 13\}$, multiplication by $\lambda = 3$, plus unity leads to the result of $\{10, 25, 40\}$. These values, for both cases, again represent the first values that are divisible by both σ_1 and the corresponding value of σ_2.

If we take $\sigma_1 = 5$ and $\sigma_2 = 7$, equation (29) leads to the following:

$$k = 7n_{2q} + \left\lceil \frac{n_{2q} - 4}{2} \right\rceil \quad \left\{n_{2q} : 5 \leq n_{2q}\right\} \tag{41}$$

This leads to the obvious question of whether or not the equation is correct. We already know that $k = 7n_{2q}$ $\{n_{2q}: 1 \leq n_{2q} \leq 4\}$. For these two speed divisions, writing out the ticks for turns $5 \leq n_{2q} \leq 13$ (the upper limit is arbitrary) shows that the turn meter is won on k = {36, 42, 51, 58, 66, 73, 81, 88, 96}. Every two turns requires an additional increment by unity. Use of equation (41) readily reproduces the tick values for both the contested and uncontested terms and without requiring calculating the preceding sequence. Using equation (33), for this case, leads to the following result:

$$k = 7n_{2q} + \frac{1}{2}\begin{pmatrix} n_{2q} - 3 - \\ \left(n_{2q} - 3\right) \bmod 2 \end{pmatrix} \quad \left\{n_{2q} : 5 \le n_{2q}\right\} \tag{42}$$

As expected, this equation also correctly replicates the values for k. Finally, for these two values, we can consider the system response based upon the use of a generating function and recursion. For $\sigma_1 = 5$ and $\sigma_2 = 7$, resulting in $\lambda = 2$, the difference between the tick values for the modified relationship when compared to the nominal relationship, starting at $n_{2j} = 5$, is the sequence {1, 1, 2, 2, 3, 3, …}. Setting these coefficients to be equal to those of A(x), we calculate $x \cdot A(x)$ as {0, 1, 1, 2, 2, 3, 3, …}. The difference $A(x) - x \cdot A(x)$ is thusly {1, 0, 1, 0, 1, 0, …}. The generating function for this sequence is $(1-x^2)^{-1}$. This leads to the following solution for A(x).

$$A(x) = \frac{1}{(1-x)(1-x^2)} \tag{43}$$

The partial fraction expansion of this equation is:

$$A(x) = \frac{1}{2}\frac{1}{(1-x)^2} + \frac{1}{4}\frac{1}{1-x} + \frac{1}{4}\frac{1}{1-(-1)x} \tag{44}$$

Using the definitions from equations (39) and (40):

$$A(n) = \frac{1}{4}\left(2n + 3 + (-1)^n\right) \tag{45}$$

This equation gives the original sequence of {1, 1, 2, 2, 3, 3, …}. We can replace n with $n_{2j} - 4$ if we wish to explicitly include n_{2j} in the formulation.

The final case shown for this section is that of $\sigma_1 = 5$ and $\sigma_2 = 9$. Here, $2 \cdot \sigma_1 = \sigma_2 + 1$. If we manually compute the values for k for $n_{2j} \ge 5$, we find that the result requires a +1 iteration for every turn. Thusly, for n_{2j} = five to ten, k = {96, 106, 116, 126, 136, 146}. Using equation (34) leads to the following solution:

$$k = 10n_{2q} - 4 \quad \left\{n_{2q} : 5 \le n_{2q}\right\} \tag{46}$$

Substitution of the turn numbers n_{2j} = five to ten produces the same result for the tick values.

B. Clan Boss

The game offers two differing opponents, currently, under the classification of Clan Boss, however, the Demon Lord Clan Boss is typically the one that is referenced when the term is used. The particular opponent in this encounter is a singular opponent and comes in six discrete tiers of difficulty that are labeled from easy to ultra nightmare (UNM). The combat statistics assigned to the opponent at each tier are known and increase with increasing difficulty. For each tier, the Clan Boss starts at neutral affinity and then changes to a non-neutral affinity once the collective damage has exceeded half of the effective health points assigned to the opponent. Once the affinity changes to one that is non-neutral, the opponent gains additional skills over the case of neutral affinity. The other salient point, here, is that at turn number 50, for the opponent, certain player combatant skills no longer work against the opponent. Other aspects of this encounter such as increased damage and immunities (outside of immunities to speed modification and direct turn meter manipulation) are not germane to this discussion. What is of relevance is that the opponent follows a fixed sequence of three different attacks with the first two consisting of area of effect (AOE) attacks and the third consisting of a single target attack.

For this example, we consider the neutral affinity case for the UNM difficulty tier encounter. The known speed of the singular opponent is 190. For each encounter, a player can utilize up to five champions. The goal, here, is not to provide a fully optimized team given that the ultimate metric is the damage generated, which in turn requires a much broader elucidation of the various skill mechanics that are available. Instead, we first consider the case of one champion with a specific skill that we seek to place in relationship to the skills used by the singular opponent. This skill is the block damage skill. When used, a block damage buff is placed on all combatants that are on a player's team. For this case, we consider the skill as being active, for two turns (until the end of the second turn), for each champion on a player's team. The two turns are predicated upon each individual champion's turns. For the champion that that possesses the ability to use the skill, the turn upon which it is used does not count as a turn for consideration in the aforementioned two turns. Thusly, if the skill is used on the champion's first turn, it remains active until the end of the champion's third turn. Two other foundational points are relevant prior to evaluation. The first is that the cooldown for the skill is four turns. Cooldowns are incremented at the start of each turn. Thusly, if the skill is used on the champion's first turn, it first becomes available, again, on the champions fifth turn. The second point is that the champion has two other skills (one with a cooldown and one without a cooldown).

For the singular opponent's speed of 190, the use of equation (2) results in the determination that the tick rate is 0.133 decimal percentage TM per tick and the use of equation (3) results in the determination that the speed division is eight. The critical turn meter value for this combatant, determinable by equation (4), is 1.064. Under nominal conditions, turn meter is won for the first AOE attack at tick k = 8, for the second AOE attack at tick k = 16 and for the single target attack at tick k = 24. This is followed by repeating the sequence at ticks 32, 40 and 48 (and so on). This sequence can be viewed as an initial transient sequence of 8-8 followed by a steady state sequence of 16-8. The three types of attacks can be encoded as $1 + 3 \cdot b_{10}$, $2 + 3 \cdot b_{10}$ and $3 + 3 \cdot b_{10}$, for the first AOE, second AOE and single target attacks, respectively, and with $\{b_{10}: b_{10} \in \mathbb{Z}_0\}$. Furthermore, for a give turn n_{1i}, the type of attack can readily be determined by equating each encoding to the turn number and determining which of the three cases produces an integer value for b_{10}. This encoding is not subject to modification by the relationship between tick and turns due to the fact that it is simply an encoding for attack type as a function of turn number.

The nominal relationship for the tick number on which turn meter is won, for each type of attack, is simply $k = 8 \cdot (1 + 3 \cdot b_{10})$, $k = 8 \cdot (2 + 3 \cdot b_{10})$ and $k = 8 \cdot (3 + 3 \cdot b_{10})$, for each type of attack, respectively. This relationship is subject to modification as a function of the critical turn meter value relative to other combatants.

For each value of b_{10}, it can be seen that the turn structure is symmetric about $b_{10} + 3$, which is the single target attack, with two antecedent AOE attacks and two subsequent AOE attacks. Thusly, each symmetric segment consists of a total of five turns with four segments. For each unit increase in b_{10}, the two subsequent AOE attacks from the previous value of b_{10} become the two antecedent AOE attacks for the incremented value of b_{10}. For the champion in question, a similar symmetric structure can be developed by using an integer encoding, b_{11} $\{b_{11}: b_{11} \in \mathbb{Z}_0\}$. Here, the turns spanned are from $b_{11} + 1$ to $b_{11} + 7$ and with the centerpoint being at $b_{11} + 4$. This holds because the effect is used at $b_{11} + 1$, expires at $b_{11} + 3$ and becomes available again at $b_{11} + 5$. The two symmetric structures can be visualized, as boxcar functions, as shown in Figure 1.

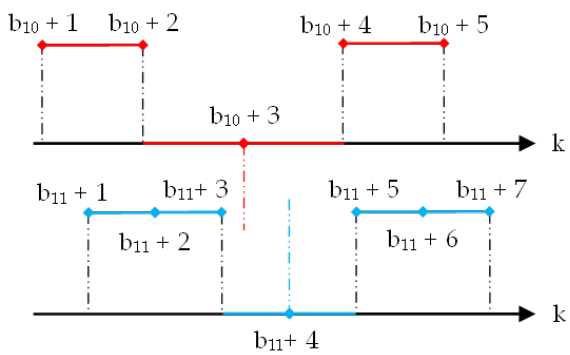

Fig. 1. Symmetric nominal turn structure as a function of tick value for the three turn sequence (red) and for the sequence consisting of a two turn effect duration with a four turn cooldown (blue)

In examining Figure 1, it can be seen that if the tick number at turn $b_{11} + 1$ is less than the tick at turn $b_{10} + 1$ and if the tick number at turn $b_{11} + 3$ is greater than the tick at turn $b_{10} + 2$, then the overlap between the symmetric sequences is maximized, for the range of values of b_{10} and b_{11}, when the difference in ticks is minimized between the points of symmetry. It is clear that the tick values for turns $b_{10} + 3$ and $b_{11} + 4$ cannot be the same as such would indicate both combatants taking a turn on the same tick value. Furthermore, one has to consider the changes that occur to the nominal relationships for the combatants as a function of relative critical turn meter. It should be noted that this approach is only one of many approaches that could be taken and is used, herein, for the purposes of this example.

For the case in which the Clan Boss was given preference at contested ticks and for no turn offsets (i.e. not using the specified skill on the first turn) for the opposing combatant, a speed division of six for the opposing combatant met the desired parameters. For each sequence, as shown in Figure 1, the difference in ticks, $(b_{11} + 4) - (b_{10} + 3)$, was unity. The difference in ticks for $(b_{11} + 1) - (b_{10} + 1)$ was negative two for the first sequence and then negative one for all subsequent sequences. The difference in ticks for $(b_{11} + 3) - (b_{10} + 2)$ was positive two for the first sequence followed by positive three for the remaining sequences. The difference in ticks for $(b_{11} + 5) - (b_{10} + 4)$ was negative one all sequences. The difference in ticks for $(b_{11} + 7) - (b_{10} + 2)$ was positive three for all sequences.

C. *Hydra*

The Hydra is the second opponent that falls under the heading of Clan Boss. This particular opponent consists of six heads, but with a constraint of only four, maximally, being present at any one time. Thusly, the encounter can be viewed as consisting of four separate opponents. This encounter is divided into six rotations. Each rotation is defined by a given set of four starting heads, each with a prescribed affinity, and with the remaining two heads, also with prescribed affinities, that are placed in a pool. Each head has prescribed skills and a prescribed set of combat statistics. The latter are known and increase with increasing difficulty (for a total of four difficulty settings). During combat, once a head is defeated (i.e. decapitated), it returns to the pool; the head that respawns is drawn from the pool. Thusly, with a total of six heads, from which four are drawn, there exist a total of 15 combinations (six choose four, which is equal to 6! divided by the multiplication of 4! times 2!). Winning an encounter for this case is defined as it was for the prior example.

For this example, we consider the starting heads for the sixth rotation and at the normal level of difficulty. In this example, the heads are referred to by name. The names, speeds [20], tick rates, speed divisions and critical turn meter values are shown in Table 4.

Table 4
Starting hydra clan boss head parameters for rotation six at normal difficulty

Head	s	t	s	TM_c
Decay	190	0.133	8	1.064
Torment	160	0.112	9	1.008
Mischief	210	0.147	7	1,029
Wrath	140	0.098	11	1.078

From Table 4, it is abundantly clear that the preference at contested ticks is in the order of Wrath, Decay, Mischief and Torment. For these four heads, Wrath represents the anchor and the relationship between which the ticks on which turns n_{1j} are won and the turn number is:

$$k = 11 n_{1j} \quad \forall n_{1j} \tag{47}$$

For Decay, since the speed divisions are not equal and with gcd(11, 8) = 1, we may readily define the relationship between the ticks on which turns n_{2j} are won and the turn number for the first ten turns.

$$k = 8 n_{2j} \quad \{n_{2j} : 1 \le n_{2j} \le 10\} \tag{48}$$

The upper limit in the previous equation derives from the fact that nominally, $k = 88$ for $n_{1j} = 8$ and also for $n_{2j} = 11$. With Wrath having precedence, the relationship is altered for Decay starting at $n_{2j} = 11$. From inspection, the equation $11n_{1j} - 8n_{2j}$

= 1, has a first solution at $n_{1j} = 3$ and $n_{2j} = 4$. Substitution of this result into equation (29) results in the following:

$$k = 8n_{2j} + \left\lceil \frac{n_{2j} - 10}{4} \right\rceil \quad \{n_2 : n_{2j} \geq 11\} \tag{49}$$

An equivalent form of equation (49), based upon equation (32), is the following:

$$k = 8n_{2j} + \frac{1}{4}\left(n_{2j} - 7 - \left(n_{2j} - 7\right) \bmod 4\right) \quad \{n_{2j} : n_{2j} \geq 11\} \tag{50}$$

While either equation (49) or (50) are directly usable, it is more instructive, for this example, to note that the relationship is one in which the number of ticks above the nominal value increases by one for every four turns inclusive of and following turn $n_{2j} = 11$. This relationship can thusly be rewritten using an integer, b_{12} $\{b_{12}: b_{12} \in \mathbb{Z}^+\}$, whose value increases by one for every turns, starting with turn 11. Thusly, the turn $n_{2j} = 50$ occurs when $k = 8n_{2j} + 10$ and at the upper limit of the domain of $47 \leq n_{2j} \leq 50$.

$$k = 8n_{2j} + b_{12} \quad \{4b_{12} + 7 \leq n_{2j} \leq 4b_{12} + 10\} \tag{51}$$

Equations (47), (48) and (51) are the patent constraint equations that are present when considering the introduction of the third head, Mischief. The speed assigned to this head places the head into speed division seven. The first constraint that would be encountered would be equation (48) secondary to the fact that $k = 56$ occurs when $n_{2j} = 7$ and when $n_{3j} = 8$. We may thus readily write the following:

$$k = 7n_{3j} \quad \{1 \leq n_{3j} \leq 7\} \tag{52}$$

For speed divisions 7 and 8, the third case presented in theory section regarding two combatants applies.We may readily define the lower limit of the inclusive domain, based upon not violating the constraint due to equation (48).

$$k = 8n_{3j} - 7 \quad \{8 \leq n_{3j} \leq \_\_\} \tag{53}$$

In equation (53), the upper limit has been left open due to the fact that it requires further consideration. In the first, we note that the constraint that arise from equation (47), at $k = 77$, specifically, is no longer valid due to the fact that the domain of the nominal form of equation (52) results in a maximum value for k of 49. This does not, by any means, however, preclude the necessity for comparing equations (47) and (53) against each other. Furthermore, the latter has to be compared with the appropriate form of equation (51). Because of the form used for equation (53), it is known that the values of k associated with the domain will not match the tick values associated with the domain of equation (51) for as long as that equation remains unchanged. The equation does change, however, from $k = 8n_{2j}$ to $k = 8n_{2j} + 1$ at turn $n_{2j} = 11$ and with $b_{12} = 1$. We may also readily see that one value for consideration as the upper limit in equation (53) is this numerical value of 11. This holds because at $n_{3j} = 12 = 11 + 1$, equation (53) gives a tick value of $8(11 + 1) - 7 = 8(11) + 1 = 89$. This is the same tick value when $n_{2j} = 11$ and thus is not correct.

Formally, for this first upper limit, we set equation (47) equal to equation (53) and set equation (51) equal to equation (53). The first equality is solved for the first value of n_{1j} that results in a tick number that is greater than that given by the lower limit of equation (53), which is 57. The second equality is solved for the first value of n_{2j} that results in a tick number that is greater than the given lower limit of equation (53). The solution that results in the lowest tick number above this lower limit is the correct solution in regards to being the next operative constraint. For both cases, the corresponding calculated value of n_{3j} must have one subtracted from it to give the correct solution for the upper boundary. This is because the calculated value of n_{3j} results in the tick numbers being equal. Starting with the second comparison, with $b_{12} = 1$, the equality is the following:

$$\begin{aligned} 8n_{2j} + 1 = 8n_{3j} - 7 &\rightarrow \\ 8n_{2j} = 8n_{3j} - 8 &\rightarrow \\ n_{2j} = n_{3j} - 1 & \end{aligned} \tag{54}$$

With $n_{2j} = 11$, $k = 88$ and the calculated value of n_{3j} is 12. Subtracting one from this result gives the expected upper limit value for this calculation of $n_{3j} = 11$. For the first case:

$$11n_{1j} = 8n_{3j} - 7 \tag{55}$$

Rewriting this equation as a congruence relationship leads to the following:

$$11n_{1j} \equiv -7 \pmod 8 \tag{56}$$

This equation is solved by first noting that $8 \cdot 5 = 40$ followed by using equation (21) and then dividing by 11 (with the notation that gcd(11, 8) = 1).

$$\begin{aligned} 11n_{1j} &\equiv -7 \pmod 8 \Rightarrow \\ 11n_{1j} &\equiv -7 + 40 \pmod 8 \Rightarrow \\ 11n_{1j} &\equiv 33 \pmod 8 \Rightarrow \\ n_{1j} &\equiv 3 \pmod 8 \end{aligned} \tag{57}$$

The final congruence shown in equation (57) translates into the following equality:

$$n_{1j} = 8q + 3 \tag{58}$$

In equation (58), $\{q: q \in \mathbb{Z}, q \geq 0\}$. At $q = 1$, $n_{1j} = 11$, which results in a value of $k = 121$ and a value of $n_{3j} = 16$. Clearly, here, the constraint generated by equation (51) is encountered first. Equation (53), with the proper upper limit, is the following:

$$k = 8n_{3j} - 7 \quad \{8 \leq n_{3j} \leq 11\} \tag{59}$$

The form of this relationship, with a lower limit of $n_{3j} = 12$, then becomes the following:

$$k = 8n_{3j} - 6 \quad \{12 \le n_{3j} \le \_\_\} \tag{60}$$

For the general case, the process for determining the upper limit from equation (53) would have to be repeated, not just for determining the proper upper limit in equation (60) but for every subsequent set of limits. For this case, however, we may first determine the set of solutions that would be apt if the constraint defined by equation (51) was the first operative constraint. We may then derive the set of solutions that would be apt if the constraint defined by equation (47) was operative, starting with the upper limit for equation (60). If the ticks predicted by each step for the latter exceed those for each step of the former, then the former is operative over all turns of interest. In examining the form of equation (59) and noting that for each step in which the constant term on the right of the equality changes that such change is by adding one, we may rewrite equation (59) in the following manner:

$$k = 8n_{3j} - 8 + b_{12} \quad \{4 + 4b_{12} \le n_{3j} \le 7 + 4b_{12}\} \tag{61}$$

It can readily be seen that the upper limit for equation (61) is equal to the lower limit for equation (51). Furthermore, for each value of p, setting the two equations equal to each other always results in the equality of $n_{2j} = n_{3j} - 1$. The values of n_{2j} are known since b_{12} is known and the lower limit of equation (51) is known in terms of b_{12}. This means that the upper limit, for each b_{12}, in equation (61) is also known. For the integer values of b_{12} from one to ten, the lower limits of equation (51) become {11, 15, 19, 23, 27, 31, 35, 39, 43, 47}. The corresponding lower limits of the same equation become {14, 18, 22, 26, 30, 34, 38, 42, 46, 50}. The lower limits of equation (61) become {8, 12, 16, 20, 24, 28, 32, 36, 40, 44}. The upper limits of the same equation become the lower limits of equation (51). The ticks for these upper limits are {81, 114, 147, 180, 213, 246, 279, 312, 345, 378}. We have already shown that the upper limit for $b_{12} = 1$ is at $n_{3j} = 11$ for the case of $b_{12} = 1$. For each successive case regarding the comparison with equation (47), we first consider the following problem:

$$11n_a \equiv 1(\mathrm{mod}\,8) \equiv 33(\mathrm{mod}\,8) \tag{62}$$

This results in a solution of n_a^{-1} mod 8 = 3 mod 8 = 3. This solution, while not the only solution, for the multiplicative modular inverse can then be used to solve for the remainder term, which becomes additive under equality, for each case of b_{12}. As an example, for the case of $b_{12} = 2$, the resulting congruence equation becomes the following:

$$11n_a \equiv -6(\mathrm{mod}\,8) \Rightarrow 11 \cdot 3n_a \equiv -6 \cdot 3(\mathrm{mod}\,8) \tag{63}$$

The first negative solution to -18 mod 8 is -2, which is congruent with 6 for this modulus. This is the same answer that can be obtained from $11n_a \equiv -6(\mathrm{mod}\ 8) \equiv -6 + 72(\mathrm{mod}\ 8) \equiv 66\ (\mathrm{mod}\ 8)$. This leads to $n_a \equiv 6\ (\mathrm{mod}\ 8)$. The first appropriate solution for $n_a = 8q + 6$ occurs at $q = 1$, which provides an answer of $n_{1j} = 14$, $k = 154$ and a solution of $n_{3j} = 20$ (which would mean an actual solution of $n_{1j} = 19$). This value of k exceeds the value, calculated from the other constraint, of 114, and thusly the constraint noted above is active at $b_{12} = 2$. Over the domain of b_{12}, performing this comparative evaluation for each value leads to tick values of {121, 154, 187, 220, 253, 286, 319, 352, 384 and 416}. Thusly, over the domain of b_{12} (i.e. from one to ten), the constraint active upon Mischief is predicated by Decay, directly, and not directly by Wrath. This leaves the fourth, and final, head, Torment, for consideration. Before including this head, note the fact that once $n_{2j} = 6$ and $n_{3j} = 7$, the ticks for the latter are equal to the ticks for the former plus one. This is inclusive for this turn and for all subsequent turns for n_{2j} and n_{3j}.

For Torment, the first constraint that impacts upon the relationship between the ticks at which turns are won is equation (48). While one might have expected it to be equation (52), such is not the case due to the fact that the speed division for Torment lies outside of the domain of the equation. Thusly,

$$k = 9n_{4j} \quad \{1 \le n_{4j} \le 7\} \tag{64}$$

As a result of the constraint, the following holds for $n_{4j} = 8$ and up to an as of yet unspecified number of turns.

$$k = 9n_{4j} + 2 \quad \{n_{4j} : 8 \le n_{4j} \le \_\_\} \tag{65}$$

Here, the increment over the nominal value is two due to the fact that the constraint equation arises from the second head, for which, within the range of ticks, the tick number immediately subsequent to the contested tick is also contested (due to the third head). An equivalent form for equation (65) is $k = 10n_{4j} - 6$ and with the domain being unchanged. For the upper limit of the domain for equation (65), we must consider equation (47), equation (51) at $b_{12} = 1$ and equation (61) at $b_{12} = 2$. This consideration leads to the first constraint as being applicable and the upper limit in equation (65) is $n_{4j} = 11$. Starting with the domain of equation (65), the limits for the resultant solution are over the inclusive turns of {8, 11}, {12, 14}, {15, 18}, {19, 21}, {22, 25}, {26, 28}, {29, 32}, {33, 35}, {36, 39} and {40, 42} for the additive terms of +2, +3, +5, +6, +8, +9, +11, +12, +14 and +15, with respect to the nominal, respectively (up through k = 393). What this shows is an alternating application of the constraint due to the second head followed by the constraint due to the first head.

The solutions developed for this example were verified using the standard approach of iterating ticks, calculating the turn meter for each combatant for each tick using the known tick rates and determining the ticks on which turns are won based upon the rules noted previously. The solution, for each head, based on this approach, matched the solutions determined previously.

4. Discussion

In the subject work, the game RSL has served the function of providing a mechanism for concretizing the theory that was presented. In this regard, it is important to note that the relationship between RSL and the theory is based, to a small extent, on information provided by the developer [16], and to a larger extent upon empirical observation and testing. No segment of the actual source code was evaluated in the preparation of this work nor was any input requested or provided by the developer. Thusly, it may be more apt to state that the game specific examples implement the underlying, presented, theory in a manner that is consistent with empirical observations and testing in regards to the game.

For the purpose of discussion, RSL may be characterized in a slightly different manner. That being one consisting of a multi-layered random number generator (RNG) coupled with a small subset of deterministic elements. If the random number generation is properly programmed to avoid any inherent bias and if there are no mechanisms present for bypassing or slanting the RNG, then the RNG component, in regards to obtaining a desired outcome for any game process, is out of the control of the player (outside of repeating the process a sufficient number of times in order to obtain the desired outcome). The consistency of the random number generation, used within the game, when compared to any ideal standard, is outside of the scope of the subject work.

Setting aside the RNG issue, it is the deterministic aspects of the game that serve as the mechanism, collectively, by which any player may impact the outcome of applicable in-game content, in a manner that is both desired and repeatable. Because all aspects of the game either directly or indirectly involve combat, and because the combat timing mechanics are predominantly deterministic, the importance of understanding the combat timing mechanics cannot be overstated. In this regard, posited in equation (2) is a simple linear relationship between two continuous variables, speed and tick rate. Within the context of abstract algebra, this relationship can be viewed as a function, which in this case is a constant, that maps elements from a domain (speed) into a codomain (tick rate). The mapping function is bijective due to the fact that each element of the codomain is mapped to be at most a single element in the domain (injective) and because each element of the codomain is mapped to by at least a single element of the domain (surjective). From the general perspective, one can trivially define any form of mapping that one wishes (e.g., split domain, non-linear, etc.).

The concept of speed divisions arises from the tick based combat timing system, the tick rate and a minimum value of turn meter at which a combatant is given consideration for winning turn meter. The speed division is simply the number of integer multiples of the tick rate required to be at or first exceed the minimum value needed for being given consideration for winning turn meter. The set of finite speed divisions, in conjunction with the tick based timing system, also generates a situation in which there will always be at least one tick value where turn meter is contested for any pair of combatants.

The speed division concept gives rise to a second mapping. The second mapping, is from tick rate to speed division. This is a mapping from a continuous domain to a discrete domain. When coupled with the rules for winning turn meter, this mapping serves to increase the complexity of the combat timing mechanics. If the set of speed divisions is chosen appropriately then each speed division is mapped to be at least one element from the set of potential tick rates. This mapping, however, is not one-to-one, due to the fact that any number of tick rate values, to the extent of the digits of precision used, map to any single speed division. The inverse mapping, that being from speed division, to tick rate, without additional constraint information, can only provide a range of values.

The concept of a critical turn meter value is one that arises from the tick based combat system, the tick rate and having a minimum value of turn meter for being consideration for winning turn meter. Equation (4) provides the relationship between speed division, tick rate and the resultant critical turn meter value. All speed divisions had an associated, inclusive, lower critical turn meter boundary value of unity. The exclusive, upper critical turn meter boundary was greater than in unity, in all cases, and with a decreasing value with increasing speed division. The resultant of a common, inclusive lower boundary, and the range of exclusive, upper boundaries, all greater than unity, is that there exist overlapping ranges of critical turn meter values, across all speed divisions. The concept of the critical turn meter value clearly shows why the approach of simply having the highest speeds is not always an appropriate approach.

The rules for winning turn meter further enhance the complexity of the mechanics. External to the factors already noted, this arises from the combination of the following: (a) allowing only one combatant to win turn meter at a given tick value and (b) allowing the turn meter of a combatant, that is above the minimum value, but not the highest value amongst all combatants, to increase further. A system with the inclusion for contemporaneous turns would actually simplify the combat mechanics.

The heavy focus on the case of two combatants, in the theoretical development presented in the subject work, was due, primarily, to the fact, that winning turn meter, for any number of combatants, involves sequential pairwise comparisons rather than simultaneous comparisons. The number of sequential comparisons required is unity for two combatants and increases by the number of extant combatants present for each additional combatant considered. These comparisons, specifically, focus on the alteration in the tick-turn relationship, from the nominal relationship, at contested tick values, for the combatant with the lower valued turn meter. In this regard, all cases in which the speeds of two combatants are not exactly equal, reduce to one of three solutions. It is only in the case where the speed divisions for the two combatants are the same must one consider the actual speeds. The solutions for the other two cases can be and were developed on the speed division level without any necessity for referencing the underlying speeds. The solution for the case of speed divisions sharing a gcd of greater than unity are given by equation (15).

The more interesting and mathematically complex case was raised by the situation in which the speed divisions are coprime. The presentation of this case in terms of the treatment of σ_1 fixed or σ_2 fixed was an artifice of presentation to the extent that each case involved varying the opposing speed division. In practice, the approach in both cases is the same, and was approached using modular arithmetic. The theory presented in this work, in regards to modular arithmetic, is not unique to RSL or games. Instead, it is operative in a number of different applications, including but not limited to certain public key cryptography systems. Returning to the context, the modular arithmetic approach involved first solving for λ, as per equation (25), and then using the value in equation (29). The form based upon using the simple mod function, given by equation (33), was simply a rearrangement of equation (29) and was provided for the sake of completeness. The theory regarding the use of generating functions and sequences, in light of the example used, which merely recreated a known sequence, might seem extraneous. The importance of the inclusion of this method is elucidated in the portion of this section that discusses future work.

The first tranche of worked examples don't invite any additional discussion. The two game related examples, on the contrary, do invite further discussion. Neither of these two examples was presented under the claim of an optimized approach. Instead, both were presented with the goal of crystallizing the underlying theory. The definition of an 'optimized' approach is multiparous rather than singular and for both cases would involve the consideration of additional combatants. For the Clan boss example, the method of using the points of symmetry for the Clan boss turn sequence and a single oppositional combatant, using a skill with a two turn duration, and a four turn cooldown, was a method only with the intent of showing the numerical relationships. The issue of whether or not this approach should or should not be within an 'optimized' approach is not addressed in the subject work. The second example presented was presented with the intent of showing the process when considering more than two combatants. The exclusion of any oppositional combatants was a purposeful choice in order to simplify the presentation and to preclude any misunderstanding regarding optimization. In regards to the first, the complexity of the tick-turn relationship should readily be appreciated by the presentation.

In regards to the Hydra example, the anchor was distally removed from the other heads, in regards to the speed division, which was taken as being the highest speed division, which corresponded to the slowest speed. An interesting aspect of the example was the interplay between the Head of Decay and the Head of Mischief. The former was at one speed division lower and with the higher critical turn meter value when compared to the latter. The numerical choices used closely couple the two heads (this makes sense in regards to the primary skills of both). The example, as a whole, while complex, only scratches the surface of the complexity of the particular encounter.

A final point of discussion is requisite, prior to considering avenues for future work. This point of discussion focuses upon one of the cited references, which is the most comprehensive written context-specific reference that was available [17]. There are a number of points that are worthy of note. None of these points detract from the significant work expressed within the reference. The issue of the 2:1 ratio of turns based upon speeds of 200 to 100 was addressed within the reference. However, the rounded value, for c, of 1430 produced an incorrect result. This rounded value was also used in the generation of a speed division table, which in turn contained boundary values that were rounded to the nearest integer value. No mention was made that the upper boundary, for each speed division, is an exclusive boundary.

The concept of a dragging effect, which was defined as '*a tactic whereby a character's speed division is dragged down by one or more*,' however, is incorrect, in regards to ubiquitous application. This does hold for the special case where $\sigma_2 + 1$ is equal to an integer multiple of σ_1. One, of course, may readily write $\sigma_i \cdot n_{ip}$ as $(\sigma_i + 1) \cdot n_{ip} - n_{ip}$, but this is a simple rewriting rather than tactically altering the speed division. The confusion may arise from equating the addition of unity to the tick value, for the losing combatant at a contested tick, to increasing the speed division by unity. Using the standard form of a linear equation, the former is $y = m \cdot x + 1$ while the latter is $y = (m + 1) \cdot x$. The reference provided substantive details on full team compositions for Clan boss battles. A discussion of the same, here, falls outside of the scope of the subject work.

Finally, it can readily be stated that the subject work is foundational. The topics covered in the subject work elucidate the basic relationships between speed, tick rate, speed divisions and turn meter. At the risk of tautology, a thorough understanding of these relationships is requisite for the proper use of the same when it comes to the subsequent applications of the same. This includes, but is not limited, to evaluating the extant conceptualizations regarding turn ratios and comprehensive team building applications [17]. There are a number of sources currently available, in the public domain, that provide information on teams that are speed tuned [21] as well as calculators for the purpose of speed tuning various in-game combat scenarios [22]-[24]. The information regarding extant, functional speed tuned teams and the calculators are invaluable resources for the player base. The calculators, however, require the entry of speed values, on the part of the user, for the combatants that they are considering for the content in question. Unless one uses external information regarding speed divisions and critical turn meter values, the approach of using speeds can lead to long rounds of trial and error due to the fact that a single incorrect speed can readily change a functional speed tune into one that fails.

One potential method for bypassing a trial and error speed based approach is to use the tick based timing system. By this, one means using the specific tick values or range of tick values at or over which one would want to have a particular combatant win turn meter. It is within this context where the theory regarding sequences and generating functions may provide for the greatest utility. In the example that was used, the sequence represented the number of ticks that were offset from the nominal case, subsequent to the first contested tick. Such an instantiation of a sequence is clearly not the only instantiation. One may readily generate a sequence consisting of tick offsets, for a given turn, with respect to an anchor or with respect to the ticks of any other combatant.

Finally, the issue of optimization is addressed. As noted previously, and perhaps noted to a greater extent than necessary, optimization is multiparous. The in-game content was divided into two categories based upon the ultimate objectives of each. These categories are not, however, mutually exclusive in regards to the ubiquitous contextual foundation of combat. This foundation inherently involves the generation of sufficient damage to the opposing combatants. Developing an optimization algorithm based on the damage mechanics would clearly require the inclusion of all salient mechanics within the structure of the algorithm. One may, however, simplify the considerations by evaluating other game facets that are deterministically related to the damage mechanics but far simpler in regards to evaluation. Given two equivalent teams of combatants, for any given in-game encounter, the team that is tuned to use the skills of the combatants in a desired and repeatable manner is far more likely to survive longer (if the skills impact on such) and deal a greater amount of damage (if the skills impact on such). From the mathematical perspective, the development of an optimization algorithm will most likely require an alternative to using an approach for continuous time systems, such as the Karush-Kuhn-Tucker (KKT) optimality conditions, followed by discretization.

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Glossary of Symbols and Variables

$\mathbb{N}$	The set of natural numbers.
$\mathbb{Z}$	The set of all integers.
$\mathbb{Z}_0^+$	The set of all positive integers and zero.
$\in$	The elements on the left are a subset of the domain on the right.
$\forall$	The statement to the left holds for all values as defined to the right.
$\equiv$	Denotes a congruence relationship.
A(x)	Series expansion of a variable in x.
N	Total number of combatants in any in-game scenario. Integer valued.
TM	Turn meter and with the subscript referencing the combatant. Continuous variable with a lower limit of zero.
TM_c	Critical value of turn meter. When a second subscript is used, it references the noted combatant. Continuous variable.
$a_{()}$	Integer valued term. The single subscript references the combatant or the order of the polynomial coefficient term associated with a power series expansion.
b	Subscripted integer valued terms with the domains defined in the body of the paper.
c	A constant with the value given in the body of the paper.
d	Greatest common integer divisor of two integers.
f(x)	A function of the independent variable x.
gcd	Greatest common divisor. Integer valued.
k	Tick count. Integer, greater than or equal to zero. When subscripted, the subscript indicates the numerical value of the instantiation of a contested turn meter (e.g. k_1 refers to the first instantiation).
lcm	Largest common multiple. Integer valued.
n	Turn number with dual subscripts. The first subscript refers to a specific combatant. When the first subscript is alphabetic, it refers to a combatant without consideration to ordering. When the first subscript is numerical, it refers to the combatant based on order as measured by the critical turn meter value. The second subscript takes two forms. When it is alphabetic it refers to the referenced term for the combatant. The second case, ( ), refers to an as of yet unknown turn number. The third case, such as (k_1), refers to the turn on which the parenthetical holds.
p	Integer valued subscript referring to turn number.
q	Integer valued subscript referring to turn number.
r	Integer valued power term for the power series expansion.
s	Speed. Continuous variable greater than or equal to zero. When subscripted refers to the speed of the subscripted combatant.
x	An independent variable.
y	A dependent variable.
α	Integer valued coefficient in a linear relationship.
β	Integer valued coefficient in a linear relationship.
γ	Integer valued coefficient in a linear relationship. Also used for partial fraction expansions.
δ	Integer valued coefficient in a linear relationship.
κ	Positive valued integer
λ	Positive valued integer.
η	Subscripted integers with the subscript referring to the combatant.
τ	Tick rate. Continuous variable greater than or equal to zero. When subscripted refers to the tick rate of the subscripted combatant.
σ	Speed division. Positive integer valued. When subscripted, if the subscript is alphabetic, it refers to a case without ordering. If the subscript is numerical, it refers to the case in which ordering is based upon critical turn meter value.
ξ	An integer valued common divisor.
ζ	Subscripted integers with the subscript referring to the combatant.